Lecture 2. Single-particle propagator in a uniform system

- General properties of the single particle Green's function.
- Free propagator in the infinite matter. Spectral functions.
- Diagrammatic rules. Examples.
- Self-energy. Dyson Equations.
- Quasi-particle approximation.
- Beyond Hartree-Fock. Second order calculation of the self-energy.

The Single particle propagator a good tool to study single particle properties

Not necessary to know all the details of the system (the full many-body wave function) but just what happens when we add or remove a particle to the system.

It gives access to all single particle properties as :

- **momentum distributions**
- **self-energy (Optical potential)**
- **effective masses**
- **spectral functions**

Also permits to calculate the expectation value of a very special two-body operator: the Hamiltonian in the ground state.

Mathematically : time-ordered propagator

$$\boxed{g(\alpha, \beta; t, t') = -\frac{i}{\hbar} \langle \Psi_0^N | T [a_{\alpha H}(t)\, a^{\dagger}_{\beta H}(t')] | \Psi_0^N \rangle}$$

* Expectation value with respect to the exact ground state of the system of N particles.
* Defined in the Heisenberg picture
* The time ordering operator T allows to consider both particle (add a particle to the system) and hole (remove a particle from the system) propagation. With the help of step functions, the time ordering operator is defined like :

$$\boxed{T\left[a_{\alpha H}(t)\, a^{+}_{\beta H}(t')\right] = \theta(t-t')\, a_{\alpha H}(t)\, a^{+}_{\beta H}(t') - \theta(t'-t)\, a^{+}_{\beta H}(t')\, a_{\alpha H}(t)}$$

* this definition includes a sign change when the two fermion operators are interchanged.
* The idea is to put operators with earlier times to the right, just to act first.

* $|\alpha\rangle$ are a suitable basis of single particle states. $|\vec{k}, m_s, m_t\rangle$ for an infinite system

$|\Psi_0^N\rangle$ is the normalized Heisenberg ground state, which usually we do not know!

$$\hat{H}\,|\Psi_0^N\rangle \; E_0^N\,|\Psi_0^N\rangle$$

In the Heisenberg picture, the single-particle operators $a_{\alpha H}(t)$ and $a^+_{\alpha H}(t)$ are

$$a_{\alpha H}(t) = e^{i\frac{\hat{H}t}{\hbar}}\, a_\alpha\, e^{-i\frac{\hat{H}t}{\hbar}}$$

$$a^+_{\alpha H}(t) = e^{\frac{i}{\hbar}\hat{H}t}\, a^+_\alpha\, e^{-\frac{i}{\hbar}\hat{H}t}$$

$$g(\alpha,\beta;\ t-t') = -\frac{i}{\hbar}\Big\{\theta(t-t')\, e^{\frac{i}{\hbar}E_0^N(t-t')}\langle\Psi_0^N|a_\alpha\, e^{-\frac{i}{\hbar}H(t-t')}a_\beta^+|\Psi_0^N\rangle$$
$$\theta(t'-t)\, e^{\frac{i}{\hbar}E_0^N(t'-t)}\langle\Psi_0^N|a_\beta^+\, e^{-\frac{i}{\hbar}H(t'-t)}a_\alpha|\Psi_0^N\rangle\Big\}$$

Interpretation for $t>t'$

$$\underset{t>t'}{g(\alpha,\beta;\, t-t')} = -\frac{i}{\hbar} \langle \Psi_0^N | e^{\frac{i}{\hbar}\hat{H}t}\, a_\alpha\, e^{-\frac{i}{\hbar}\hat{H}t}\, e^{\frac{i}{\hbar}\hat{H}t'}\, a_\beta^\dagger\, e^{-\frac{i}{\hbar}\hat{H}t'} |\Psi_0^N\rangle$$

$e^{-\frac{i}{\hbar}\hat{H}t'} |\Psi_0^N\rangle$ ground state at t'

$a_\beta^\dagger e^{-\frac{i}{\hbar}\hat{H}t'} |\Psi_0^N\rangle$ a particle in the state β is added a $t=t'$

$e^{-\frac{i}{\hbar}\hat{H}(t-t')} a_\beta^\dagger e^{-\frac{i}{\hbar}\hat{H}t'} |\Psi_0^N\rangle$ the system evolves up to t.

$$\Big(\underbrace{a_\alpha^\dagger e^{-\frac{i}{\hbar}\hat{H}t} |\Psi_0^N\rangle}_{\text{one particle in the state } \alpha \text{ adde at time } t}\Big)^\dagger = \langle \Psi_0^N | e^{\frac{i}{\hbar}\hat{H}t}\, a_\alpha$$

therefore, for $t > t'$, $g(\alpha, \beta; t-t')$ gives the probability amplitude to find the system at time t with an additional particle in the state $|\alpha\rangle$ when at time $t' < t$ a particle in the state $|\beta\rangle$ was added to the system.

Inserting the identity of the $N+1$ and $N-1$ particle systems in the definition of $g(\alpha, \beta; t-t')$

$$g(\alpha, \beta; t-t') = -\frac{i}{\hbar}\Big\{\theta(t-t') \sum_m e^{\frac{i}{\hbar}(E_0^N - E_m^{N+1})(t-t')} \langle \Psi_0^N | a_\alpha | \Psi_m^{N+1}\rangle \langle \Psi_m^{N+1} | a_\beta^+ | \Psi_0^N \rangle$$

$$- \theta(t'-t) \sum_n e^{\frac{i}{\hbar}(E_0^N - E_n^{N-1})(t'-t)} \langle \Psi_0^N | a_\beta^+ | \Psi_n^{N-1}\rangle \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle \Big\}$$

* The propagator depends only on the time difference
→ make a Fourier transform

* We have used the knowledge of the spectrum of the system with N+1 and N-1 particles

$$\hat{H} |\Psi_m^{N+1}\rangle = E_m^{N+1} |\Psi_m^{N+1}\rangle$$

$$\hat{H} |\Psi_n^{N-1}\rangle = E_n^{N-1} |\Psi_n^{N-1}\rangle$$

$$g(\alpha,\beta;E) = \int_{-\infty}^{\infty} d(t-t')\; e^{\frac{i}{\hbar}E(t-t')}\; g(\alpha,\beta;t-t') =$$

$$= \int_{-\infty}^{\infty} d(t-t')\, e^{\frac{i}{\hbar}E(t-t')} \Big[-\frac{i}{\hbar}\Big\{\theta(t-t')\sum_m e^{\frac{i}{\hbar}(E_0^N - E_m^{N+1})(t-t')}$$

$$\langle \Psi_0^N | a_\alpha | \Psi_m^{N+1}\rangle\langle \Psi_m^{N+1} | a_\beta^+ | \Psi_0^N\rangle$$

$$-\theta(t'-t)\sum_n e^{\frac{i}{\hbar}(E_0^N - E_n^{N-1})(t'-t)} \langle \Psi_0^N | a_\beta^+ | \Psi_n^{N-1}\rangle\langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N\rangle\Big\}\Big]$$

introducing the integral representation of the step function

$$= \int_{-\infty}^{\infty} d(t-t')\; e^{\frac{i}{\hbar}E(t-t')} \Big[-\frac{i}{\hbar}\Big\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE' \frac{e^{\frac{-i}{\hbar}E'(t-t')}}{E'+i\eta^+}$$

$$\sum_m e^{\frac{i}{\hbar}(E_0^N - E_m^{N+1})(t-t')} \langle \Psi_0^N | a_\alpha | \Psi_m^{N+1}\rangle\langle \Psi_m^{N+1} | a_\beta^+ | \Psi_0^N\rangle$$

$$-\Big(-\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE'\, e^{-\frac{i}{\hbar}E'(t'-t)}\Big)\sum_n e^{\frac{i}{\hbar}(E_0^N - E_n^{N-1})(t'-t)} \langle \Psi_0^N | a_\beta^+ | \Psi_n^{N-1}\rangle\langle \Psi_n | a_\alpha | \Psi_0^N\rangle\Big\}\Big]$$

changing the order of integration

$$- (-i) \int_{-\infty}^{\infty} \frac{dE'}{2\pi i} \int_{-\infty}^{\infty} d\left(\frac{t-t'}{\hbar}\right) \sum_m \frac{e^{-i\left[E' - (E_0^N - E_m^{N+1}) - E\right]\frac{(t-t')}{\hbar}}}{E' + i\eta^+} \langle \Psi_0^N | a_\alpha | \Psi_m^{N+1} \rangle \langle \Psi_m^{N+1} | a_\beta^+ | \Psi_0^N \rangle$$

$$+ (-i) \int_{-\infty}^{\infty} \frac{dE'}{2\pi i} \int_{-\infty}^{\infty} d\left(\frac{t-t'}{\hbar}\right) \sum_n \frac{e^{-i\left[-E' + (E_0^N - E_n^{N-1}) - E\right]\frac{(t-t')}{\hbar}}}{E' + i\eta^+}$$

$$\langle \Psi_0^N | a_\beta^+ | \Psi_n^{N-1} \rangle \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle$$

$$= - (-i) \int_{-\infty}^{\infty} \frac{dE'}{2\pi i} 2\pi \sum_m \frac{\delta(E' - (E_0^N - E_m^{N+1}) - E)}{E' + i\eta^+} \langle \Psi_0^N | a_\alpha | \Psi_m^{N+1} \rangle \langle \Psi_m^{N+1} | a_\beta^+ | \Psi_0^N \rangle$$

$$- i \int \frac{dE'}{2\pi i} 2\pi \sum_n \frac{\delta(-E' - (E_0^N - E_n^{N-1}) - E)}{E' + i\eta^+} \langle \Psi_0^N | a_\beta^+ | \Psi_n^{N-1} \rangle \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle$$

The presence of the δ allows to perform the integral on dE', and one gets

$$g(\alpha,\beta;E) = \sum_m \frac{\langle \Psi_0^N | a_\alpha | \Psi_m^{N+1} \rangle \langle \Psi_m^{N+1} | a_\beta^+ | \Psi_0^N \rangle}{E + E_0^N - E_m^{N+1} + i\eta^+} - \sum_n \frac{\langle \Psi_0^N | a_\beta^+ | \Psi_n^{N-1} \rangle \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle}{-E + E_0^N - E_n^{N-1} + i\eta^+}$$

and finally

$$g(\alpha,\beta;E) = \sum_m \frac{\langle \Psi_0^N | a_\alpha | \Psi_m^{N+1} \rangle \langle \Psi_m^{N+1} | a_\beta^+ | \Psi_0^N \rangle}{E + E_0^N - E_m^{N+1} + i\eta^+} + \sum_n \frac{\langle \Psi_0^N | a_\beta^+ | \Psi_n^{N-1} \rangle \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle}{E - (E_0^N - E_n^{N-1}) - i\eta^+}$$

This is the Lehmann representation

Removing the complete set of $|N+1\rangle$ eigenstates and $|\Psi_n^{N-1}\rangle$ one expresses the Green function as an expectation value on the ground state

$$g(\alpha,\beta;E) = \langle \Psi_0^N | a_\alpha \frac{1}{E-(\hat{H}-E_0^N)+i\eta} a_\beta^\dagger | \Psi_0^N \rangle$$

$$+ \langle \Psi_0^N | a_\beta^\dagger \frac{1}{E-(E_0^N-\hat{H})-i\eta} a_\alpha | \Psi_0^N \rangle$$

Occupation of the single-particle state $|\alpha\rangle$

$$n(\alpha) = \langle \Psi_0^N | a_\alpha^\dagger a_\alpha | \Psi_0^N \rangle = \sum_n \left| \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle \right|^2$$

$$= \int_{-\infty}^{\varepsilon_F} dE \sum_n \left| \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle \right|^2 \delta\left(E - (E_0^N - E_n^{N-1})\right)$$

$$= \int_{-\infty}^{\varepsilon_F} dE \, S_h(\alpha, E) = \frac{1}{\pi} \int_{-\infty}^{\varepsilon_F} dE \, \mathrm{Im} \, g(\alpha, \alpha; E)$$

$$= \oint \frac{dE}{2\pi i} \, g(\alpha, \alpha; E)$$

Several ways
to calculate $n(\alpha)$

Similar procedure for the dis occupation

$$d(\alpha) = \langle \Psi_0^N | a_\alpha a_\alpha^+ | \Psi_0^N \rangle = \sum_m' |\langle \Psi_m^{N+1} | a_\alpha^+ | \Psi_0^N \rangle|^2$$

$$= \int_{\varepsilon_F}^{\infty} dE \sum_m' |\langle \Psi_m^{N+1} | a_\alpha^+ | \Psi_0^N \rangle|^2 \, \delta(E - (E_m^{N+1} - E_0^N))$$

$$= \int_{\varepsilon_F}^{\infty} dE \; S_p(\alpha, E) = -\frac{1}{\pi} \int_{\varepsilon_F}^{\infty} dE \; \mathrm{Im}\, g(\alpha, \alpha; E)$$

$$n(\alpha) + d(\alpha) = 1 = \langle \Psi_0^N | a_\alpha^+ a_\alpha + a_\alpha^+ a_\alpha | \Psi_0^N \rangle$$

$$= \langle \Psi_0^N | \{a_\alpha^+, a_\alpha\} | \Psi_0^N \rangle = 1$$

The occupation + disoccupation $= 1$

The Green's function has simple poles at the excitation energies of the (A−1) system (hole-part), and of the (A+1) system (particle-part)

Complex E-plane

hole-part
Poles above the real axis

particle-part
Poles below the real axis

For a continuous excitation spectrum becomes a cut

Complex E-plane

$$g(\alpha,\alpha;E)=\sum_m{}' \frac{\langle\Psi_0^N|a_\alpha|\Psi_m^{N+1}\rangle\langle\Psi_m^{N+1}|a_\alpha^\dagger|\Psi_0^N\rangle}{E+E_0^N-E_m^{N+1}+i\eta^+}$$

$$+\sum_n{}' \frac{\langle\Psi_0^N|a_\alpha^\dagger|\Psi_n^{N-1}\rangle\langle\Psi_n^{N-1}|a_\alpha|\Psi_0^N\rangle}{E-E_0^N+E_n^{N-1}-i\eta^+}$$

use $\frac{1}{E\pm i\eta}=P\frac{1}{E}\mp i\pi\delta(E)$ $\Rightarrow$

$E<\cancel{\mu}=E_0^N-E_0^{N-1}$

$$S_h(\alpha,E)=\frac{1}{\pi}\operatorname{Im} g(\alpha,\alpha,E)$$

$$=\sum_n{}' |\langle\Psi_n^{N-1}|a_\alpha|\Psi_0^N\rangle|^2\,\delta(E-(E_0^N-E_n^{N-1}))$$

$S_h(\alpha,E)$ gives the probability distribution to take out a particle in the state $|\alpha\rangle$ from the ground state $|\Psi_0^N\rangle$ leaving the resulting N-1 system ~~with~~ in a state $|\Psi_n^{N-1}\rangle$ with energy $E_n^{N-1}=E_0^N-E$

$$\boxed{\hat{O}_1 = \sum_{\alpha\beta}' \langle\alpha|O_1|\beta\rangle\, a^\dagger_\alpha a_\beta}$$

Expectation value of $\hat{O}_1$ in the ground state:

$$\langle\Psi_0^N|\hat{O}_1|\Psi_0^N\rangle = \sum_{\alpha\beta}' \langle\alpha|O|\beta\rangle\, n_{\alpha\beta}$$

where $n_{\alpha\beta} = \langle\Psi_0^N|a^\dagger_\alpha a_\beta|\Psi_0^N\rangle$

How is related $n_{\alpha\beta}$ and $g(\alpha,\beta;E)$?

$$\boxed{\begin{aligned} n_{\beta\alpha} &= \oint \frac{dE}{2\pi i}\, g(\alpha,\beta;E) \\ &= \int \frac{dE}{2\pi i}\, e^{iE\eta}\, g(\alpha,\beta;E) \end{aligned}}$$

Integral in the complex E plane.

The convergence factor $e^{iE\eta}$ forces to close the contour in the upper part of the complex E plane

Remember the Lehmann representation

$$g(\alpha,\beta;E) = \sum_m' \frac{\langle\Psi_0^N|a_\alpha|\Psi_m^{N+1}\rangle\langle\Psi_m^{N+1}|a^\dagger_\beta|\Psi_0^N\rangle}{E-(E_m^{N+1}-E_0^N)+i\eta}$$

poles in the lower part

$$+\sum_n' \frac{\langle\Psi_0^N|a^\dagger_\beta|\Psi_n^{N-1}\rangle\langle\Psi_n^{N-1}|a_\alpha|\Psi_0^N\rangle}{E-(E_0^N-E_n^{N-1})-i\eta}$$

poles in the upper part

What are the residues? Pole of order 1

$$\lim_{E\to(E_0^N-E_n^{N-1})+i\eta} g(\alpha,\beta;E)\,\left(E-(E_0^N-E_n^{N-1})-i\eta\right)$$

$$= \langle\Psi_0^N|a^\dagger_\beta|\Psi_n^{N-1}\rangle\langle\Psi_n^{N-1}|a_\alpha|\Psi_0^N\rangle$$

$$\oint \frac{dE}{2\pi i}\, g(\alpha,\beta;E) = 2\pi i \frac{1}{2\pi i} \sum \mathrm{Res} =$$

$$= \sum_n \langle \Psi_0^N | a_\beta^+ | \Psi_n^{N-1} \rangle \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle$$

$$= \langle \Psi_0^N | a_\beta^+ a_\alpha | \Psi_0^N \rangle = n_{\beta\alpha}$$

and as $S_h(\alpha,E) = \frac{1}{\pi} \operatorname{Im} g(\alpha,\alpha;E) =$

$$= \sum_n |\langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle|^2 \, \delta(E-(E_0^N - E_n^{N-1}))$$

we can write

$$\oint \frac{dE}{2\pi i}\, g(\alpha,\alpha;E) = \frac{1}{\pi}\int_{-\infty}^{\varepsilon_F} dE \operatorname{Im} g(\alpha,\alpha;E)$$

$$= \int_{-\infty}^{\varepsilon_F} S_h(\alpha,E)\, dE = n_{\alpha\alpha}$$

The Green's function has simple poles at the excitation energies of the (A−1) system (hole-part), and of the (A+1) system (particle-part)

Complex E-plane

hole-part
Poles above the real axis

particle-part
Poles below the real axis

For a continuous excitation spectrum becomes a cut

Complex E-plane

Free Fermi gas

All k occupied up to k_F

$$\rho = \frac{\nu}{(2\pi)^3} \int d^3k \; \Theta(k_F - k)$$

ν spin-isospin degeneracy

$|\phi_0^N\rangle \Rightarrow$ Slater-determinant of plane waves

$= a^+$

$$\hat{H} = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} a^+_{\vec{k}} a_{\vec{k}}$$

No spin labels

$$\hat{H} |\phi_0^N\rangle = E_0^N |\phi_0^N\rangle$$

$$\hat{H} a^+_{\vec{k}} |\phi_0^N\rangle = \left(E_0 + \frac{\hbar^2 k^2}{2m}\right) a^+_{\vec{k}} |\phi_0^N\rangle \qquad k > k_F$$

$k < k_F$

$$= 0$$

$$\hat{H} a_{\vec{k}} |\phi_0^N\rangle = \left(E_0 - \frac{\hbar^2 k^2}{2m}\right) a_{\vec{k}} |\phi_0^N\rangle \qquad k < k_F$$

$k > k_F$

$$= 0$$

$$g^{(0)}(k,E) = \sum_m \frac{\langle \Psi_0^N | a_{\vec{k}} | \Psi_m^{N+1}\rangle\langle \Psi_m^{N+1} | a_{\vec{k}}^{+} | \Psi_0^N\rangle}{E - (E_m^{N+1} - E_0^N) + i\eta} +$$

$$+ \sum_n \frac{\langle \Psi_0^N | a_{\vec{k}}^{+} | \Psi_n^{N-1}\rangle\langle \Psi_n^{N-1} | a_{\vec{k}} | \Psi_0^N\rangle}{E - (E_0^N - E_n^{N-1}) - i\eta}$$

there is contribution from only one intermediate state $\Rightarrow$

$$= \left\{ \frac{\theta(k-k_F)}{E - \left(\left(E_0^N + \frac{\hbar^2 k^2}{2m}\right) - E_0^N\right) + i\eta} + \frac{\theta(k_F - k)}{E - \left(E_0^N - \left(E_0^N - \frac{\hbar^2 k^2}{2m}\right)\right) - i\eta} \right\}$$

$$g^{(0)}(k,E) = \frac{\theta(k-k_F)}{E - \frac{\hbar^2 k^2}{2m} + i\eta} + \frac{\theta(k_F-k)}{E - \frac{\hbar^2 k^2}{2m} - i\eta}$$

To calculate the spectral functions

$$\frac{1}{A \pm i\eta} = \mathcal{P}\left(\frac{1}{A}\right) \mp i\pi\,\delta(A)$$

then

$$\underset{E<\varepsilon_F}{S_h(k,E)} = \frac{1}{\pi}\,\mathrm{Im}\,g^{(0)}(k,E) = \delta\left(E - \frac{\hbar^2 k^2}{2m}\right)\theta(k_F - k)$$

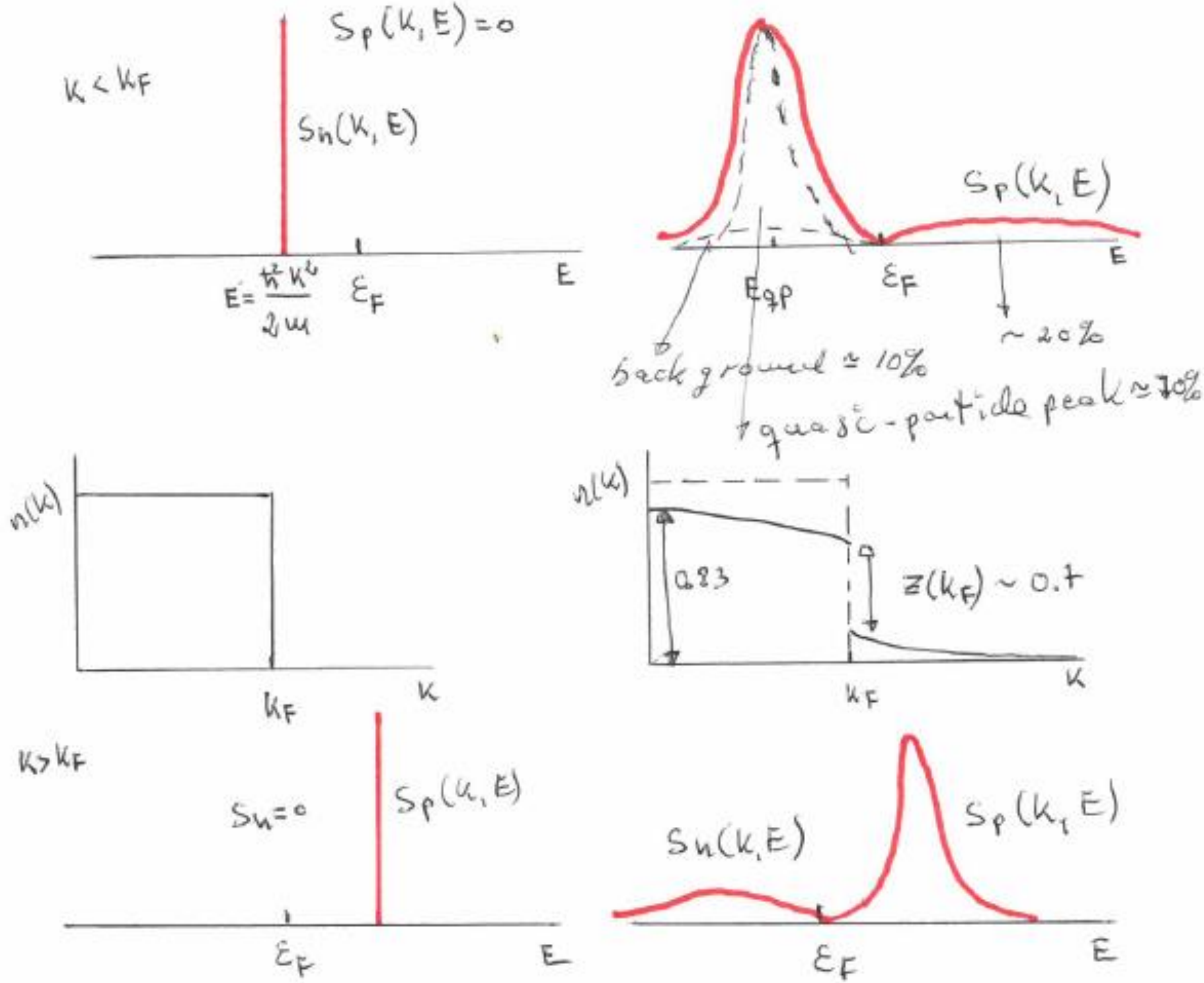
$$\underset{E>\varepsilon_F}{S_p(k,E)} = -\frac{1}{\pi}\,\mathrm{Im}\,g^{(0)}(k,E) = \delta\left(E - \frac{\hbar^2 k^2}{2m}\right)\theta(k - k_F)$$

The momentum distribution

$$n(k) = \int_{-\infty}^{\varepsilon_F} S_h(k,E)\,dE = \theta(k_F - k)\int_{-\infty}^{\varepsilon_F} \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) dE = \theta(k_F - k)$$

$$d(k) = \int_{-\infty}^{\varepsilon_F} S_p(k,E)\,dE = \theta(k - k_F)$$

$$\Rightarrow n(k) + d(k) = 1$$


k < kF
Sp(k,E)=0
Sh(k,E)
E = ħ²k²/2m
εF
E
Sp(k,E)
Eqp
εF
E
~ 20%
background ≃ 10%
quasi-particle peak ≃ 70%
n(k)
kF
k
n(k)
0.83
z(kF) ~ 0.7
kF
k
k>kF
Sh=0
Sp(k,E)
εF
E
Sh(k,E)
Sp(k,E)
εF
E

Free Fermi gas. Kinetic energy and Koltun sum-rule

$$\frac{1}{N}\langle \hat{T}\rangle_{FS} = \frac{1}{N}\frac{V}{(2\pi)^3}\overset{\text{spin-isospin}}{\nu}\int d^3k \,\underbrace{\langle \vec{k}|t|\vec{k}\rangle}_{\frac{\hbar^2k^2}{2m}}\,\underbrace{\int_{-\infty}^{\varepsilon_F} S_h(k,E)\,dE}_{\theta(k_F-k)}$$

$$= \frac{1}{\rho}\frac{1}{(2\pi)^3}\,\nu\int d^3k\,\frac{\hbar^2k^2}{2m}\,\theta(k_F-k) = \frac{1}{\rho}\frac{\nu}{(2\pi)^3}\left[\frac{\hbar^2k_F^2}{2m}\right]4\pi\frac{1}{5}k_F^3$$

$$= \frac{3}{5}\,\frac{\hbar^2k_F^2}{2m}$$

$$\frac{1}{N}\langle \hat{H}\rangle_{FS} = \frac{1}{\rho}\frac{\nu}{(2\pi)^3}\,\frac{1}{2}\int d^3k\int_{-\infty}^{\varepsilon_F} dE\left(\frac{\hbar^2k^2}{2m}+E\right)\underbrace{S_h(k,E)}_{\delta\left(E-\frac{\hbar^2k^2}{2m}\right)\theta(k_F-k)}$$

$$= \frac{1}{\rho}\frac{\nu}{(2\pi)^3}\,\frac{1}{2}\int d^3k\cdot 2\,\frac{\hbar^2k^2}{2m} = \frac{3}{5}\,\frac{\hbar^2k_F^2}{2m}$$

For uniform system and continuous spectrum. Normal Fermi system, no pairing.

Lehmann representation

$$g(k, E) = \int_{-\infty}^{\varepsilon_F} dE' \frac{S_h(k, E')}{E - E' - i\eta} + \int_{\varepsilon_F}^{\infty} dE' \frac{S_p(k, E')}{E - E' + i\eta}$$

$$= \int_{-\infty}^{\varepsilon_F} \left\{ P \frac{S_h(k, E')}{E - E'} + i\pi\, \delta(E - E')\, S_h(k, E') \right\} dE'$$

$$+ \int_{\varepsilon_F}^{\infty} \left\{ P \frac{S_p(k, E')}{E - E'} - i\pi\, \delta(E - E')\, S_p(k, E') \right\} dE' =$$

$$\operatorname{Im} \underset{E < \varepsilon_F}{g(k, E)} = \pi \int_{-\infty}^{\varepsilon_F} \delta(E - E')\, S_h(k, E')\, dE' = \pi\, S_h(k, E)$$

and for $E > \varepsilon_F$ we have:

$$\operatorname{Im} \underset{E > \varepsilon_F}{g(k, E)} = -\pi \int_{\varepsilon_F}^{\infty} \delta(E - E')\, S_p(k, E')\, dE' = -\pi\, S_p(k, E)$$

we can separate $g(k, E)$ in two pieces

We can separate $g(k,E)$ in two pieces:

$$g_p(k,E) = \int_{\varepsilon_F}^{\infty} dE' \frac{S_p(k,E')}{E-E'+i\eta} \qquad g_n(k,E) = \int_{-\infty}^{\varepsilon_F} dE' \frac{S_n(k,E')}{E-E'-i\eta}$$

$$\frac{i}{2\pi}\Big[g_p(k, E+i\delta) - g_p(k, E-i\delta)\Big]_{\delta \to 0^+,\ \delta \geq 0}$$

$$= \frac{i}{2\pi}\left[\int_{\varepsilon_F}^{\infty} dE' \frac{S_p(k,E')}{E+i\delta-E'+i\eta} - \int dE' \frac{S_p(k,E')}{E-i\delta-E'+i\eta}\right]$$

In the limit $\eta \to 0$
I can always consider $|\delta| >$

$$= \frac{i}{2\pi}\left[P\int_{\varepsilon_F}^{\infty} dE' \frac{S_p(k,E')}{E-E'} - i\pi S_p(k,E) - P\int_{\varepsilon_F}^{\infty} dE' \frac{S_p(k,E')}{E-E'} - i\pi S_p(k,E)\right] = S_p(k,E)$$

$$\boxed{\underset{E \geq E_F}{S_p(k,E)} = \frac{i}{2\pi}\Big[g_p(k,E+i\delta) - g_p(k,E-i\delta)\Big]}$$

$$\langle \hat{H} \rangle = \frac{1}{2} \sum_{\bar{k}} \int_{-\infty}^{\varepsilon_F} dE \left(\frac{\hbar^2 k^2}{2m} + E \right) S_h(k, E)$$

if we want the energy per particle,

$$\frac{1}{N} \sum_{\bar{k}} \longrightarrow \frac{1}{(2\pi)^3 \rho} \int d^3k$$

$$\frac{1}{N} \langle \hat{H} \rangle = \frac{1}{2} \frac{1}{(2\pi)^3 \rho} \int d^3k \int_{-\infty}^{\varepsilon_F} dE \left(\frac{\hbar^2 k^2}{2m} + E \right) S_h(k, E)$$

in the ground state

no spin-isospin is included

$\boxed{\text{Perturbative expansion of the time evolution operator in the interaction picture}}$

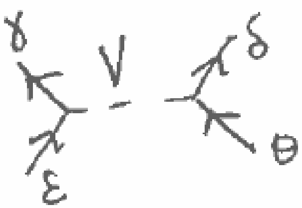
$\boxed{\text{Diagrammatic rules}}$

$\boxed{\text{Wick's theorem to evaluate expectation values}}$

Assume

$$\boxed{\begin{aligned}\hat{H} &= \sum_{\alpha} \langle \alpha | T | \alpha \rangle a_\alpha^{+} a_\alpha \\ &+ \frac{1}{2} \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\, a_\alpha^{+} a_\beta^{+} a_\delta a_\gamma\end{aligned}}$$

Different diagrammatic elements



$\Rightarrow \quad \langle \gamma\,\delta \,|\, V \,|\, \varepsilon\,\theta \rangle$

left in

left out   right out   right in

For a uniform system we take plane waves normalized to volume

$$\frac{1}{\Omega^2}\int d^3r_1\, d^3r_2\; e^{-i\bar{k}_\gamma \bar{r}_1}\, e^{-i\bar{k}_\delta \bar{r}_2}\, V(r_{12})\, e^{i\bar{k}_\varepsilon \bar{r}_1}\, e^{i\bar{k}_\theta \bar{r}_2}$$

* At a given order "n" we draw n horizontal lines, and two external points a^+_β and a_α.

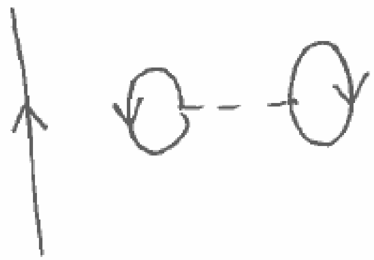
at the bottom   at the top

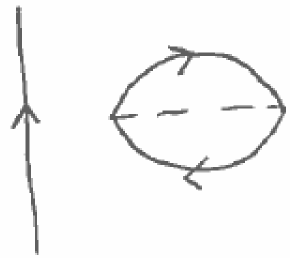
* Now one should joint the lines without finding arrows against ⇒ perform the contractions of Wick's theorem properly!.

* At order "n" ⟶ there are n interactions
V ⇒ We have to join $2n+2$ points
⇒ $2n+1$ lines ⇒ $(2n+1)$ g_0 factors!

* At order "n" we have $4n+2$ creation and annihilation operators
⇒ $(2n+1)$ creation
$(2n+1)$ annihilation
⇒ In principle I can make $(2n+1)!$ terms all contracted

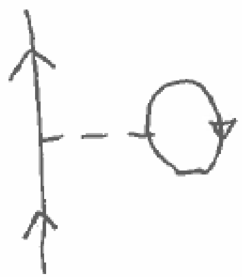
Order $n = 1$

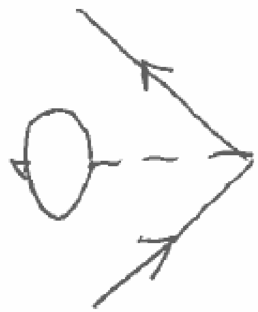




"unlinked".

* One needs to separate the pen from the paper.
* Cancelled with the denominator.





"topologically equivalent"

Calculate only one!

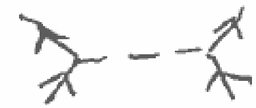
"topologically equivalent"

At order n appear (for each linked diagram) 2^n topologically equivalent diagrams. We calculate only one and cancel the factor $\frac{1}{2^n}$ which appear at order "n" due to the potential.

Rules for ig. Order n

(1) Draw all topologically different diagrams with $\underline{n}$ interaction lines and $2n+1$ $g^{(0)}$ propagators.

to this end:

a) Draw "n" horizontal lines  with vertices "in" and "out" arrows plus the two external points

b) Join all vertices starting from the lower one with an in-going and an out-going line at each vertex.

- To visualize if two diagrams are equivalent ⇒ try to deform them and see if they look the same.

- These are Feynman diagrams ⇒ The arrows in the lines indicate flux of energy and momentum, which are conserved at each vertex.

- The diagrams contain particles and holes

(2) Assign a label $\vec{k}, m_\sigma, E$ at each line (also m_t in nuclear systems!)

At each line $(g^{(0)})$ corresponds a factor:

$$i\, g^{(0)}_{m_\sigma m'_\sigma}(k,E) = i\, \delta_{m_\sigma m'_\sigma} \left[\frac{\theta(|\vec{k}|-k_F)}{E-\frac{\hbar^2 k^2}{2m}+i\eta} + \frac{\theta(k_F-|\vec{k}|)}{E-\frac{\hbar^2 k^2}{2m}-i\eta} \right]$$

which contains propagation of particles and holes.

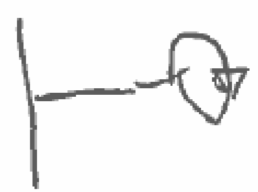
(3) Assign a matrix element for each interaction. Momentum and energy are conserved at each vertex

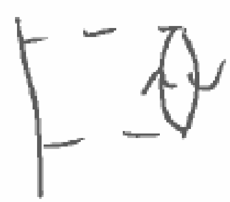
$$(-i)\ \langle \bar{k}_1 \bar{k}_2 | V(r) | \bar{k}_3 \bar{k}_4 \rangle$$

(4) Sum over al internal quantum numbers. spin and momenta, $\Omega \int \frac{d^3k}{(2\pi)^3}$

and energies $\int \frac{dE}{2\pi}$

(5) Assign a sign $(-1)^F$ where F is the number of Fermionic loops ⇒ fermionic lines that close over themselves



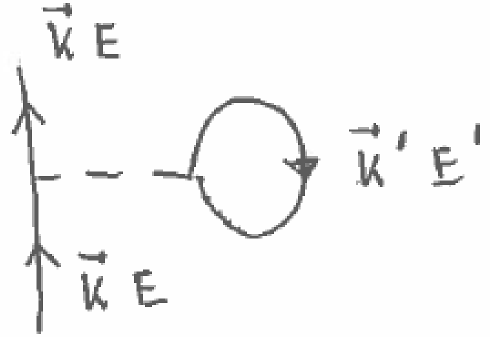


1 fermionic loop

6. The integral over the energy of a Fermionic non propagating line (starts and ends at the same interaction) should be performed in the upper part of the complex plane

$$\oint \frac{dE}{(2\pi)}\, g^{(0)}(k, E)$$

(which is in agreement with the receipt for contractions at equal times).



First order diagram

6 operators

3 $g^{(0)}$

linked diagram

* We have taken into account the momentum and energy conservation at each vertex.
* Let's forget the spin and the isospin.

Contribution to $i g(k, E)$

i^3 one for each $g^{(0)}$ $\quad (-i)$ one interaction $\quad (-1)$ fermionic loop $\quad g^{(0)}(k, E)$. $\langle \vec{k}\vec{k}' | V | \vec{k}\vec{k}' \rangle \, g^{(0)}(k, E)$

$$\sum_{\vec{k}'} \frac{1}{2\pi} \int dE' \, g^{(0)}(k', E')$$

$$\langle \bar{k}\,\bar{k}' | V | \bar{k}\,\bar{k}' \rangle = \frac{1}{\Omega} \int d^3r\, V(r)$$

normalization to volume

$$= g^{(0)}(k,E) \Big[\underbrace{\frac{\cancel{\Omega}}{(2\pi)^3} \int d^3k'}_{\sum_{k'}} \; \frac{1}{2\pi} \oint dE'\, g^{(0)}(k,E') \; \frac{1}{\cancel{\Omega}} \int d^3r\, V(r) \Big]\; g^{(0)}(k,E)$$

$$\frac{1}{2\pi} \oint dE' \left[\frac{\theta(k'-k_F)}{E'-\varepsilon_{k'}+i\eta} + \frac{\theta(k_F-k')}{E'-\varepsilon_{k'}-i\eta} \right] = \frac{1}{2\pi}\, 2\pi i \sum \mathrm{Res} = i\,\theta(k_F-k')$$

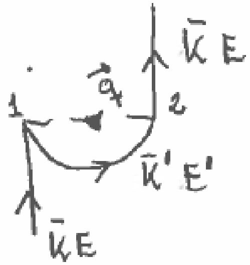
$$\operatorname*{Res}_{E'\to\varepsilon_{k'}+i\eta} g^{(0)}(k',E') = \lim_{E'\to\varepsilon_{k'}+i\eta} (E'-\varepsilon_{k'}-i\eta)\, g^{(0)}(k',E') = \theta(k_F-k')$$

$$= i\, g^{(0)}(k,E) \Big[\underbrace{\frac{1}{(2\pi)^3} \int d^3k'\, \theta(k_F-k')}_{\rho} \int d^3r\, V(r) \Big]\, g^{(0)}(k,E)$$

$$= i\, g^{(0)}(k,E)\, \rho \int d^3r\, V(r)\, g^{(0)}(k,E) = i\, g^{(1)}(k,E)$$

dimensions?

$$\left.\begin{aligned} \left[g^{(0)}(k,E)\right] &= \frac{1}{E} \\ \left[\rho \int d^3r\, V(r)\right] &= E \end{aligned}\right\} \frac{1}{E}\, E\, \frac{1}{E} = \frac{1}{E}$$



Contribution to $i\, g(k,E)$

$$i^3 \quad (-i) \quad g^{(0)}(k,E) \sum_{\vec k'} \frac{1}{2\pi} \oint g^{(0)}(k',E')\, dE'$$

$$\langle \bar k' \, \bar k \,|\, v(r) \,|\, \bar k \, \bar k' \rangle \; g^{(0)}(k,E)$$

leftout rightout leftin rightin

$$\langle \bar k' \, \bar k \,|\, v(r) \,|\, \bar k \, \bar k' \rangle = \frac{1}{\Omega^2} \int d^3 r_1 \int d^3 r_2 \; e^{-i \bar k' \bar r_1} e^{-i \bar k \bar r_2} \, v(r_{12}) \, e^{i \bar k \bar r_1} e^{i \bar k' r_2}$$

$$= \frac{1}{\Omega^2} \int d^3 R \int d^3 r \; e^{-i \bar k' (\bar R + \frac{\bar r}{2})} e^{-i \bar k (\bar R - \frac{\bar r}{2})} \, v(r) \, e^{i \bar k (\bar R + \frac{\bar r}{2})} e^{i \bar k' (\bar R - \frac{\bar r}{2})}$$

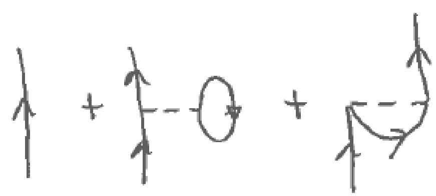
$$= \underbrace{\frac{1}{\Omega} \int d^3 R}_{1} \cdot \frac{1}{\Omega} \int d^3 r \; e^{-i (\bar k' - \bar k) \frac{\bar r}{2}} \, v(r) \, e^{i (\bar k - \bar k') \frac{\bar r}{2}}$$

$$= \frac{1}{\Omega} \int d^3 r \; v(r) \, e^{-i \bar q \bar r} \qquad \bar q = \bar k' - \bar k$$

$$\frac{1}{2\pi}\oint dE'\, g^{(0)}(k',E') = i\,\theta(k_F-k')$$

every thing together ⇒

$$= -i\, g^{(0)}(k,E)\left[\frac{1}{(2\pi)^3}\int d^3k'\,\theta(k_F-k')\int d^3r\,\vartheta(r)\,e^{-i(\vec{k}'-\vec{k})r}\right] g^{(0)}(k,E)$$



$$\Rightarrow\; i\,g^{(0)}(k,E) +$$

$$i\,g^{(0)}(k,E)\left[\rho\int d^3r\,\vartheta(r) - \frac{1}{(2\pi)^3}\int d^3k'\,\theta(k_F-k')\int d^3r\,\vartheta(r)\,e^{-i(\vec{k}'-\vec{k})\vec{r}}\right] g^{(0)}(k,E)$$

At the moment it is not easy to identify the excitation energies as poles in the denominators ⇒ We need to do a Dyson equation ⇒ an infinite sum of diagrams

Self-energy and Dyson equation

The analysis of the diagrams contributing to the one-body Green's function allows to introduce the concept of self-energy.

Self-energy: Any diagram without external legs, wich can be inserted into a propagator line.

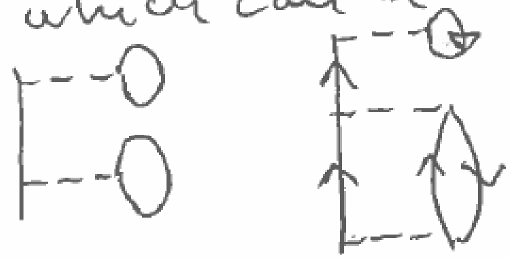
Examples:

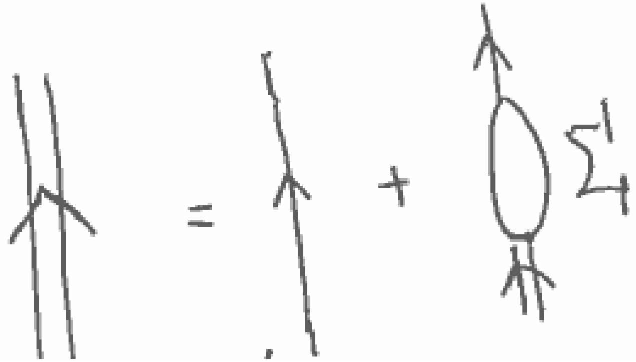
Proper (**irreducible**) self-energy:

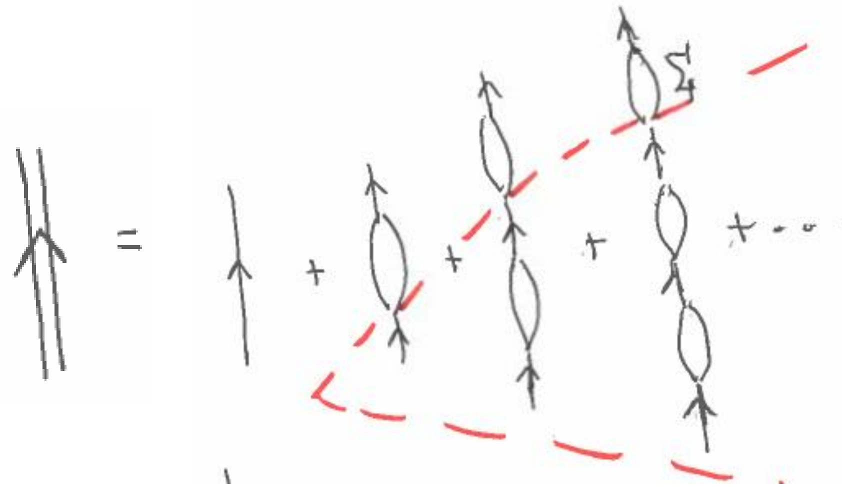
A self-energy part which can not be broken into two unconnected self-energy parts by removing one propagator line.

Reducible

A self-energy part which can be broken







Dyson equation

$$i\,g(k,E) = i\,g^{(0)}(k,E) + i\,g^{(0)}(k,E)\,(-i)\Sigma(k,E)\;i\,g(k,E)$$

$$g(k,E) = g^{(0)}(k,E) + g^{(0)}(k,E)\,\Sigma(k,E)\,g(k,E)$$

$\Sigma(k,E)$ is a complex object.

$$g(k,E) = g^{(0)}(k,E) + g^{(0)}(k,E)\, \Sigma_1(k,E)\, g(k,E$$

is an algebraic equation

$$g(k,E) - g^{(0)}(k,E)\, \Sigma_1(k,E)\, g(k,E) = g^{(0)}(k,E$$

$$g(k,E) = \frac{g^{(0)}(k,E)}{1 - g^{(0)}(k,E)\, \Sigma_1(k,E)} = \frac{1}{g^{(0)}(k,E)^{-1} - \Sigma}$$

$$\boxed{g(k,E) = \frac{1}{E - \frac{\hbar^2 k^2}{2m} - \Sigma_1(k,E)}}$$

$\Sigma_1(k,E)$ complex object related to the interaction of the particle with

$$g(k,E) = \frac{1}{E - \frac{\hbar^2 k^2}{2m} - \Sigma_R(k,E) - i\,\Sigma_I(k,E)}$$

$$= \frac{E - \frac{\hbar^2 k^2}{2m} - \Sigma_R(k,E) + i\,\Sigma_I(k,E)}{\left[E - \frac{\hbar^2 k^2}{2m} - \Sigma_R(k,E)\right]^2 + \left[\Sigma_I(k,E)\right]^2}$$

for $\forall k$

$$S_h(k,E) = \frac{1}{\pi}\,\mathrm{Im}\, g(k,E) \quad \text{for } E < \epsilon_F$$

$$S_h(k,E) = \frac{1}{\pi}\,\frac{\Sigma_I(k,E)}{\left[E - \frac{\hbar^2 k^2}{2m} - \Sigma_R(k,E)\right]^2 + \left[\Sigma_I(k,E)\right]^2}$$

$$\underline{S_h(k,E) > 0 \Rightarrow \Sigma_I(k,E) > 0 \quad E < \epsilon_F}$$

$$E > \varepsilon_F$$

$$S_p(k,E) = -\frac{1}{\pi}\, \mathrm{Im}\, g(k,E)$$

$$S_p(k,E) = -\frac{1}{\pi}\, \frac{\Sigma_I(k,E)}{\left[E - \frac{\hbar^2 k^2}{2m} - \Sigma_R(k,E)\right]^2 + \left[\Sigma_I(k,E)\right]^2}$$

$$E > \varepsilon_F$$

$$S_p(k,E) > 0 \Rightarrow \Sigma_I(k,E) < 0$$

$$\Sigma_I(k,E) > 0 \qquad E < \varepsilon_F \qquad \Rightarrow \forall k \quad \Sigma_I(k,\varepsilon_F) = 0$$

$$\Sigma_I(k,E) < 0 \qquad E > \varepsilon_F$$

Quasi-particle approximation

$$g(k,E) = \frac{1}{E - \frac{\hbar^2 k^2}{2m} - \Sigma^1(k,E)}$$

Expanding the real part of $\Sigma^1(k,E)$ around $\varepsilon(k)$

$$\varepsilon(k) = \frac{\hbar^2 k^2}{2m} + U(k)$$

$$U(k) \equiv \mathrm{Re}\, \Sigma^1(k, \varepsilon(k))$$

$$W(k) \equiv \mathrm{Im}\, \Sigma^1(k, \varepsilon(k))$$

$$g_{QP}(k,E) = \frac{1}{E - \frac{\hbar^2 k^2}{2m} - U(k) - \left.\frac{\partial \operatorname{Re}\Sigma'_1}{\partial E}\right|_{E=\varepsilon(k)} (E-\varepsilon(k)) - iW}$$

$$= \frac{1}{E - \varepsilon(k)\left(1 - \left.\frac{\partial \operatorname{Re}\Sigma'_1}{\partial E}\right|_{E=\varepsilon(k)}\right) - iW}$$

$$= \frac{\left(1 - \left.\frac{\partial \operatorname{Re}\Sigma'_1(k,E)}{\partial E}\right|_{E=\varepsilon(k)}\right)^{-1}}{E - \varepsilon(k) - i\left(1 - \left.\frac{\partial \operatorname{Re}\Sigma'_1}{\partial E}\right|_{E=\varepsilon(k)}\right)^{-1} W}$$

$$Z(k) = \left\{ 1 - \frac{\partial \mathrm{Re} \Sigma_1(k,E)}{\partial E} \right\}^{-1}_{E=\varepsilon(k)}$$

strength of the quasi-particle pole

$$g_{qp}(k,E) = \frac{Z(k)\,(E-\varepsilon(k))}{(E-\varepsilon(k))^2 + (Z(k)\,W(k))^2} + i\,\frac{Z^2(k)\,W}{(E-\varepsilon(k))^2+(Z\,W)^2}$$

Its invers Fourier transform, represents the probability amplitude that one particle of momentum k propgates like a dumped plane wave with frequency E(k) and lifetime $\tau = (2Z(k)W(k))^{-1}$

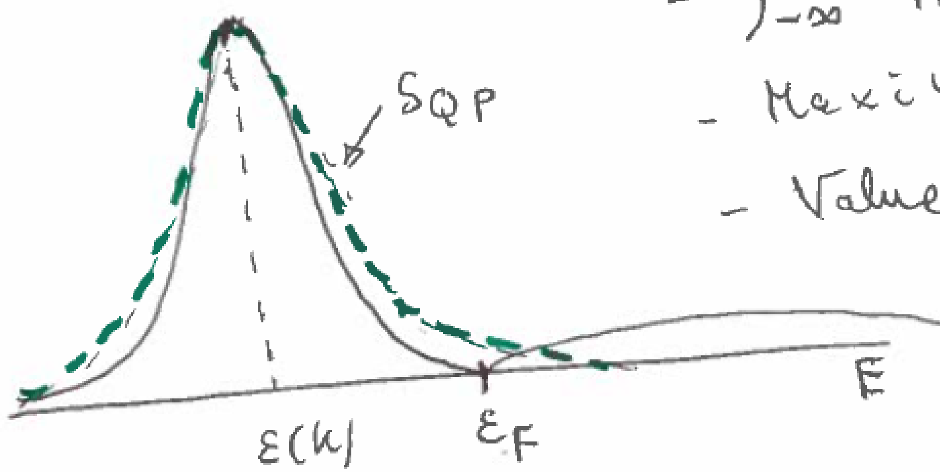
$$g_{qp}(k,t) = -iZ(k)e^{-iE(k)t}e^{-Z(k)W(k)t}$$

$$S_{qp}(k,E) = \frac{1}{\pi} \frac{Z^2(k)\,|W|}{(E-\varepsilon(k))^2 + (Z(k)\,W(k))^2}$$

Lorentziana

- $\int_{-\infty}^{\infty} S_{qp}(k,E)\, dE = Z(k)$
- Maximum at $\varepsilon(k)$
- Value at the maximum

$$\frac{1}{\pi} \frac{1}{\operatorname{Im} \Sigma'(k, \varepsilon(k))}$$



- Width.

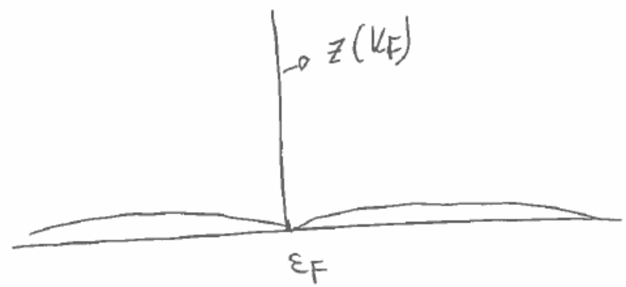
$$\frac{1}{2}\frac{1}{\pi}\frac{1}{W} = \frac{1}{\pi}\frac{Z^2 W}{(E-\varepsilon)^2 + Z^2 W^2}$$

$$\Rightarrow \quad E - \varepsilon(k) = Z \cdot W$$

The width is $2Z \cdot W$

$$\Sigma'_I (k, E = \varepsilon_F) = 0 \quad \forall k$$

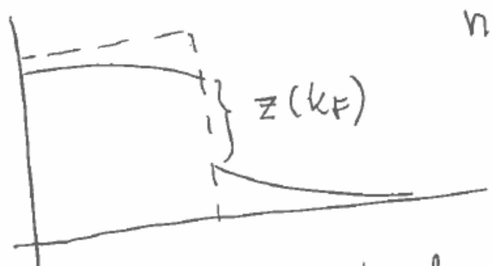
$$\to \Sigma'_I (k_F, \varepsilon_F) = 0 \quad \text{in this case.}$$



the S_{qp} becomes a $\delta(E - \varepsilon_F)$ with strength $Z(k_F)$

That illustrates that the discontinuity of $n(k)$ at k_F is given by $Z(k_F)$

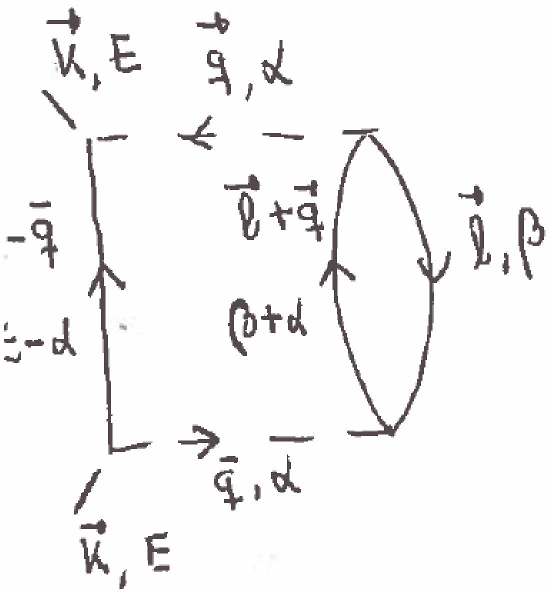
$$n(k) = \int_{-\infty}^{\varepsilon_F} dE \; S_h(k, E)$$



The quasi-particle peak for $k > k_F$, appears at $E > \varepsilon_F$ and therefore does not contribute to $n(k)$!

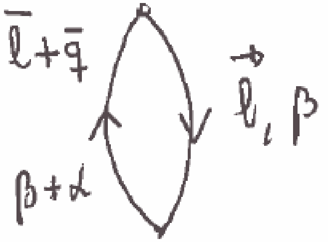
Second order diagram

Not antisymmetric matrix elements



I consider the diagram as a contribution to the self-energy (I remove the external legs!)

Remember! The rules for $-i\Sigma(k, E)$ are the same as for $ig(k, E)$



$$= \underset{(-1)}{\overset{\text{fermionic loop}}{}} \; 2 \; \frac{1}{(2\pi)^3} \int d^3 l \int_{-\infty}^{\infty} \frac{d\beta}{2\pi}$$

$$i \left[\overset{①}{\frac{\theta(l - k_F)}{\beta - \varepsilon(l) + i\delta}} + \overset{②}{\frac{\theta(k_F - l)}{\beta - \varepsilon(l) - i\delta}} \right]$$

$$i \left[\overset{③}{\frac{\theta(|\vec{l}+\vec{q}| - k_F)}{\beta + \alpha - \varepsilon(|\vec{l}+\vec{q}|) + i\delta}} + \overset{④}{\frac{\theta(k_F - |\vec{l}+\vec{q}|)}{\beta + \alpha - \varepsilon(|\vec{l}+\vec{q}|) - i\delta}} \right]$$

Now I perform the integral over β.

I have 4 combinations

① × ③ Does not contribute, because they have the poles in the same half complex plane.

② × ④ The same argument applies in this case.

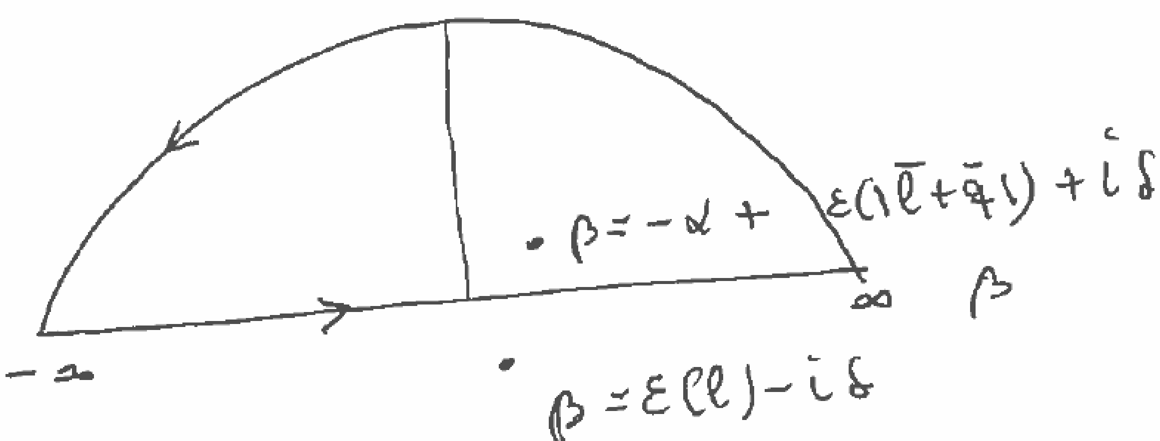
We have contribution from cases that mixes particle and hole having the poles in different half-planes!

Case ① x ④, we have poles at

$\beta = \varepsilon(l) - i\delta$ and $\beta = -\alpha + \varepsilon(|\bar{l}+\bar{q}|) + i\delta$

below   above

We close the contour in the upper part!



$$\oint = \oint + \int_{-\infty}^{\infty} = 2\pi i \sum_i \mathrm{Res}\,.$$

but the contribution from $\oint \propto \lim\limits_{R\to\infty} \int \frac{d(Re^{i\theta})}{2\pi} \frac{i}{Re^{i\theta}} \frac{i}{Re^{i\theta}}$

$$\propto \lim_{R\to\infty} \frac{1}{R} \to 0$$

$$\int_{-\infty}^{\infty} \frac{d\beta}{2\pi}\, i^2\, \frac{\theta(l-k_F)}{\beta-\varepsilon(l)+i\delta}\; \frac{\theta(k_F-|\bar{l}+\bar{q}|)}{\beta+\alpha-\varepsilon(|\bar{l}+\bar{q}|)-i\delta}$$

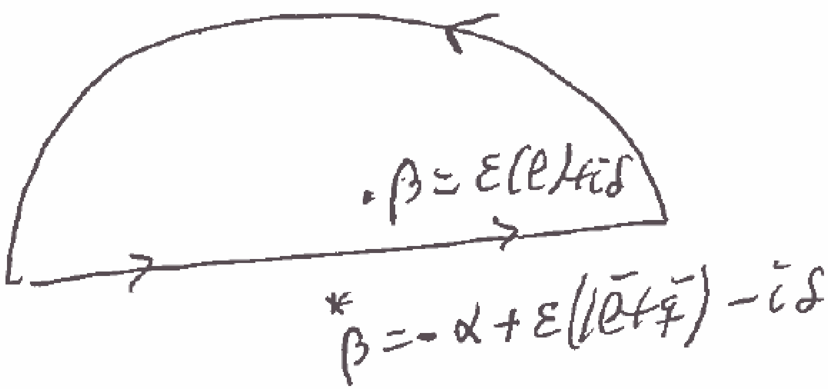
$$= i^2\, \frac{\cancel{2}\pi i}{\cancel{2\pi}} \lim_{\beta\to-\alpha+\varepsilon(|\bar{l}+\bar{q}|)+i\delta} \frac{\theta(l-k_F)}{\beta-\varepsilon(l)+i\delta}\; \frac{\theta(k_F-|\bar{l}+\bar{q}|)}{\cancel{(\beta+\alpha-\varepsilon(|\bar{l}+\bar{q}|)-i\delta)}}\; \cancel{(\beta-(-\alpha+\varepsilon(|\bar{l}+\bar{q}|)+i\delta))}$$

$$= -i\, \frac{\theta(l-k_F)\;\theta(k_F-|\bar{l}+\bar{q}|)}{-\alpha+\varepsilon(|\bar{l}+\bar{q}|)+i\delta-\varepsilon(l)+i\delta}$$

$$\boxed{= \frac{i\;\theta(l-k_F)\;\theta(k_F-|\bar{l}+\bar{q}|)}{\alpha-\varepsilon(|\bar{l}+\bar{q}|)+\varepsilon(l)-i\delta}}$$

Contribution ② × ③

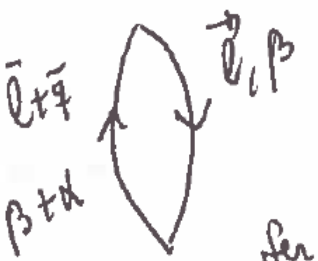
$$\int_{-\infty}^{\infty}\frac{d\beta}{2\pi}\, i^2\, \overset{\text{hole}}{\frac{\theta(k_F-l)}{\beta-\varepsilon(l)-i\delta}}\; \overset{\text{particle}}{\frac{\theta(|\bar{l}+\bar{q}|-k_F)}{\beta+\alpha-\varepsilon(|\bar{l}+\bar{q}|)+i\delta}} =$$



$$= i^2\,\frac{2\pi i}{2\pi}\lim_{\beta\to\varepsilon(l)+i\delta}\frac{\cancel{\beta-\varepsilon(l)-i\delta}}{\cancel{\beta-\varepsilon(l)-i\delta}}\;\frac{\theta(k_F-l)\,\theta(|\bar{l}+\bar{q}|-k_F)}{\beta+\alpha-\varepsilon(|\bar{l}+\bar{q}|)+i\delta}$$

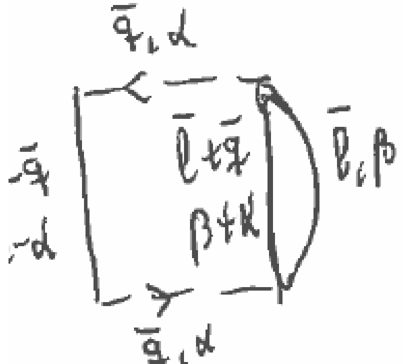
$$= -i\,\frac{\theta(k_F-l)\quad\theta(|\bar{l}+\bar{q}|-k_F)}{\varepsilon(l)+\alpha-\varepsilon(|\bar{l}+\bar{q}|)+i\delta}$$

Therefore, now we perform the integration over momenta



$$= \underbrace{(-1)}_{\text{fermionic loop}}\ \underbrace{2}_{\text{sum over spin}} \left[\frac{1}{(2\pi)^3} \int d^3l \, \frac{i\,\theta(l-k_F)\,\theta(k_F-|\bar{l}+\bar{q}|)}{\alpha-\varepsilon(|\bar{l}+\bar{q}|)+\varepsilon(l)-i\delta} - \frac{1}{(2\pi)^3} \int d^3l \, \frac{i\,\theta(k_F-l)\,\theta(|\bar{l}+\bar{q}|-k_F)}{\alpha+\varepsilon(l)-\varepsilon(|\bar{l}+\bar{q}|)+i\delta} \right]$$

Now we add the integration over d^3q and over $d\alpha$. Consider just the term corresponding to $\theta(k_F-\ell)\,\theta(|\bar\ell+\bar q|-k_F)$



two interactions

propagator associated to $\bar k-\bar q$

$$= \underbrace{(-i)^2}_{\text{two interactions}}\ \underbrace{i}_{\text{propagator}}\ \underbrace{2}_{\text{spin}}\ \underbrace{i}_{\text{ph. propagator}}$$

$$\frac{1}{(2\pi)^3}\int d^3q\ V_q^2\ \frac{1}{(2\pi)^3}\int d^3\ell \int \frac{d\alpha}{2\pi}\left[\frac{\theta(|\bar k-\bar q|-k_F)}{E-\alpha-\varepsilon(|\bar k-\bar q|)+i\delta}+\frac{\theta(k_F-|\bar k-\bar q|)}{E-\alpha-\ldots-i\delta}\right]$$

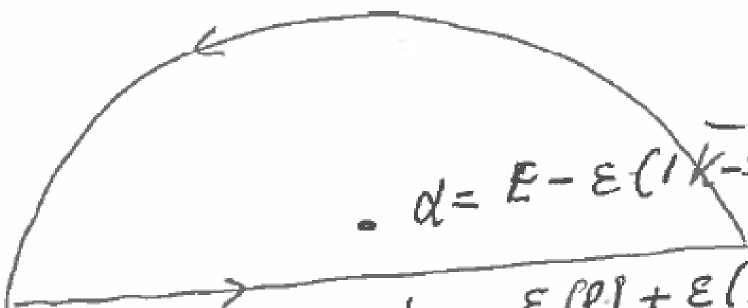
new energy integration

$$\frac{\theta(k_F-\ell)\ \theta(|\bar\ell+\bar q|-k_F)}{\alpha+\varepsilon(\ell)-\varepsilon(|\bar\ell+\bar q|)+i\delta}$$

$+$ ... tribution

When I integrate over $d\alpha$ I get con
only from the particle part of the propagator
associated to $\bar{k}-\bar{q}$.

pole associated to $g^{(0)}(|\bar{k}-\bar{q}|, E-\alpha)$
particle part



$\alpha = E - \varepsilon(|\bar{k}-\bar{q}|) + i\delta$

$\alpha = -\varepsilon(l) + \varepsilon(|\bar{l}+\bar{q}|) - i\delta$ Pole associated to the ph propagator

The result of the integral =

$$\frac{2\pi i}{2\pi} \lim_{\alpha \to E-\varepsilon(|\bar{k}-\bar{q}|)+i\delta} \underbrace{\frac{\alpha - E + \varepsilon(|\bar{k}-\bar{q}|) - i\delta}{E-\alpha-\varepsilon(|\bar{k}-\bar{q}|)+i\delta}}_{-1} \frac{1}{\alpha + \varepsilon(l) - \varepsilon(|\bar{l}+\bar{q}|) + i\delta}$$

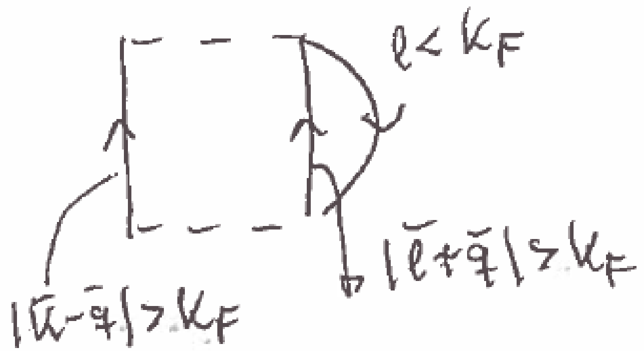
$$= -i \, \frac{\theta(k_F - l)\,\theta(|\bar{l}+\bar{q}|-k_F)\,\theta(|\bar{k}-\bar{q}|-k_F)}{E+\varepsilon(l)-\varepsilon(|\bar{l}+\bar{q}|)-\varepsilon(|\bar{k}-\bar{q}|)+i\delta}$$

Therefore the contribution of this piece (2p1h) :

$$-i\,\Sigma^{(2p1h)}(k,E) = (-i)^2\, 2\, i^3 (-1) \frac{1}{(2\pi)^3}\int d^3q\, \frac{1}{(2\pi)^3}\int d^3l$$

$$|V_q|^2 \;\theta$$

$$= -2i \int \frac{d^3q}{(2\pi)^3}\int \frac{d^3l}{(2\pi)^3}\; |V_q|^2 \; \frac{\theta(k_F-l)\,\theta(|\vec{l}+\vec{q}|-k_F)\,\theta(|\vec{k}-\vec{q}|-k_F)}{E+\varepsilon(l)-\varepsilon(|\vec{l}+\vec{q}|)-\varepsilon(|\vec{k}-\vec{q}|)+i\delta}$$



Goldstone diagram

$$\mathrm{Im}\,\Sigma^{2p1h} = -2\pi \int\frac{d^3q}{(2\pi)^3}\int\frac{d^3l}{(2\pi)^3}\,|V_q|^2\,\delta\big(E+\varepsilon(l)-\varepsilon(|\bar{l}+\bar{q}|)-\varepsilon(|\bar{k}-\bar{q}|)\big)$$

spin

associated to δ

* The imaginary part of Σ^{2p1h} is negative
* Contribution for $E>\varepsilon_F$ which is the minimum value of E that produces imaginary part for a given k

$$E = \varepsilon(|\bar{l}+\bar{q}|) - \varepsilon(l) + \varepsilon(|\bar{k}-\bar{q}|)$$

$l < k_F \quad |l+q| > k_F \quad |\bar{k}-\bar{q}| > k_F$

Assume $\varepsilon(k)$ an increasing function of k

$$\text{min of } E = \min\left(\varepsilon(|\bar{l}+\bar{q}|)\right) - \max \varepsilon(l) + \min(|\bar{k}-\bar{q}|)$$

$$= \varepsilon_F$$

$$\Rightarrow \quad \mathrm{Im}\,\Sigma^{2p1h}(k,E) = 0 \qquad E < \varepsilon_F$$

$$< 0 \qquad E > \varepsilon_F$$

So, the single-particle momentum gets mix with 2p1h configuration.

One can show that $\mathrm{Im}\,\Sigma^{2p1h}(k,E)$ close to the Fermi surface behaves like

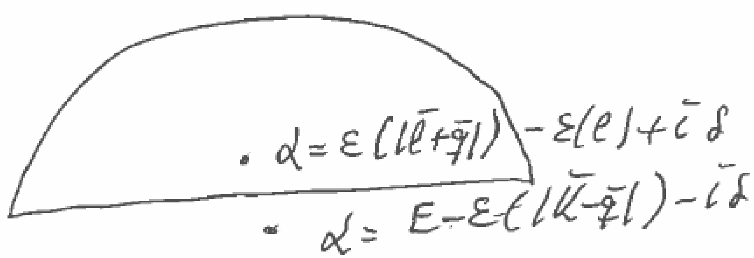
$$\mathrm{Im}\,\Sigma^{2p1h}(k,E) \propto (E-\varepsilon_F)^2 \quad E > \varepsilon_F$$

Let's look at the other contribution, that we will call **2h1p**

$$(-i)^2 2i(-i) \int \frac{d^3q}{(2\pi)^3} V_q^2 \int \frac{d^3l}{(2\pi)^3} \int \frac{d\alpha}{2\pi}$$

$$\frac{\theta(l-k_F)\,\theta(k_F-|\vec{l}+\vec{q}|)}{\alpha-\varepsilon(|\vec{l}+\vec{q}|)+\varepsilon(l)-i\delta}\left[\frac{\theta(|\vec{k}-\vec{q}|-k_F)}{E-\alpha-\varepsilon(|\vec{k}-\vec{q}|)+i\delta}+\frac{\theta(k_F-|\vec{k}-\vec{q}|)}{E-\alpha-\varepsilon(|\vec{k}-\vec{q}|)-i\delta}\right]$$

we get contribution only from the hole part of the propagator associated to $\vec{k}-\vec{q}$.



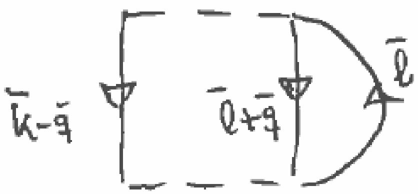
$$\frac{\cancel{2\pi i}}{\cancel{2\pi}} \lim_{\alpha \to \varepsilon(|\vec{l}+\vec{q}|)-\varepsilon(l)+i\delta} \underbrace{\frac{\alpha-\varepsilon(|\vec{l}+\vec{q}|)+\varepsilon(l)-i\delta}{\alpha-\varepsilon(|\vec{l}+\vec{q}|)+\varepsilon(l)-i\delta}}_{1}$$

$$\frac{\theta(l-k_F)\,\theta(k_F-|\vec{l}+\vec{q}|)\,\theta(k_F-|\vec{k}-\vec{q}|)}{E-\alpha-\varepsilon(|\vec{k}-\vec{q}|)-i\delta}$$

$$= i\,\frac{\theta(l-k_F)\,\theta(k_F-|\vec{l}+\vec{q}|)\,\theta(k_F-|\vec{k}-\vec{q}|)}{E-\varepsilon(|\vec{l}+\vec{q}|)+\varepsilon(l)-\varepsilon(|\vec{k}-\vec{q}|)-i\delta}$$

$$-i\,\Sigma^{2h1p}(k,E) = -2i\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3l}{(2\pi)^3}\,V_q^2\,\frac{\theta(l-k_F)\,\theta(k_F-|\vec{l}+\vec{q}|)\,\theta(k_F-|\vec{k}-\vec{q}|)}{E+\varepsilon(l)-\varepsilon(|\vec{l}+\vec{q}|)-\varepsilon(|\vec{k}-\vec{q}|)-i\delta}$$

$$\mathrm{Im}\, \Sigma^{12h1p} = 2\pi \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3l}{(2\pi)^3} |V_q|^2 \, \delta\big(E - \varepsilon(l) - \varepsilon(|\bar{l}+\bar{q}|) - \varepsilon(|\bar{k}-\bar{q}|)\big)$$



$|\bar{l}+\bar{q}| < k_F$

$|k-q| < k_F$

$l > k_F$

$$\mathrm{Im}\, \Sigma^{12h1p}(k,E) > 0$$

One gets imaginary part for $E < \varepsilon_F$

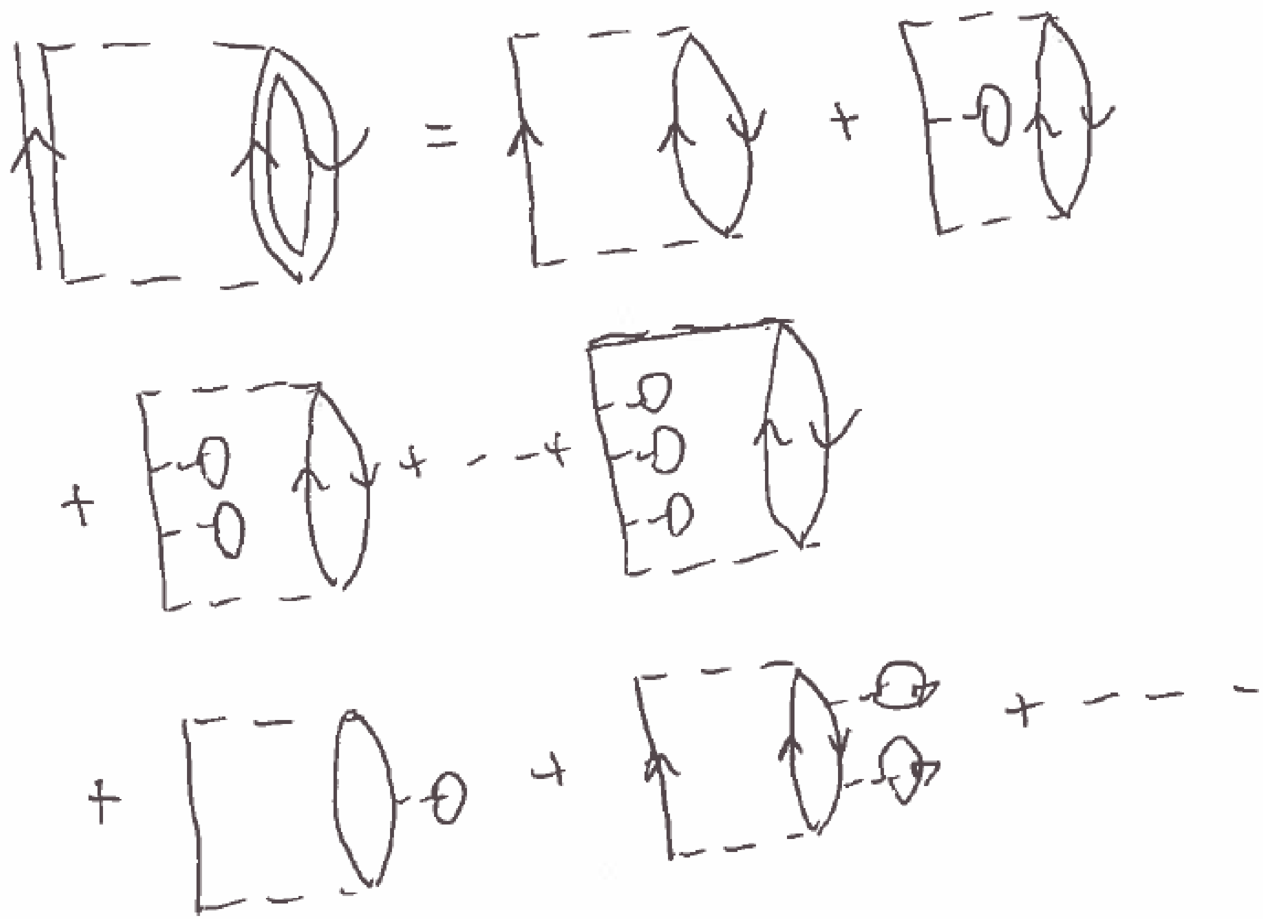
$$E = \varepsilon(|\bar{l}+\bar{q}|) + \varepsilon(|\bar{k}-\bar{q}|) - \varepsilon(l)$$

$l > k_F$ $\quad |\bar{k}-\bar{q}| < k_F \quad |l+q| < k_F$

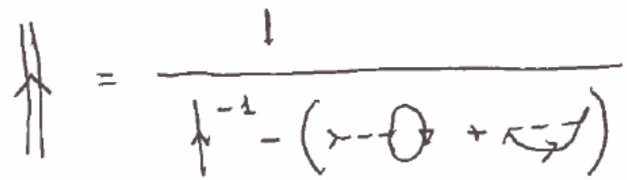
Max E ?

$= \varepsilon_F$

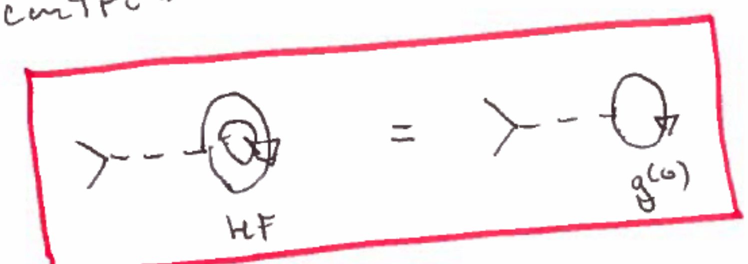
Dressing the lines (with care!)



Hartree - Fock



If we recalculate the self-energy diagrams with a the HF level I get the same contribution.



because

$$\boxed{\begin{aligned} \oint dE\; g^{(0)}(k,E) &= 2\pi i\,\theta(k_F-k) \\ \oint dE\; g^{HF}(k,E) &= 2\pi i\,\theta(k_F-k) \end{aligned}}$$

Next lecture

Now we are ready to include the ladder diagrams in the self-energy, to take Care of the short-range repulsion present in the NN interaction.

We will include both propagation of particles and holes at all orders.
Dressing the intermediate states in the T-matrix with the full spectral functions
This defines a self-consistent problema between the determination of the scattering of the dressed particles and the dressing of the particles.

On one side the interaction affects the properties of the particles : dressing through the self-energy (spectral functions) and at the same time the dressing modifies the effective interaction between the dressed particles.

The mínimal consistent approximation is to consider the ladder approach.

Propagating only particles and under certain approaches for the intermediate propagators we can recover the BHF approach.