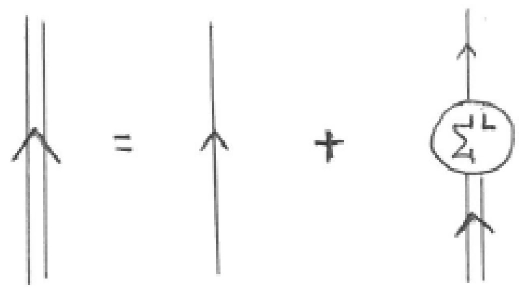
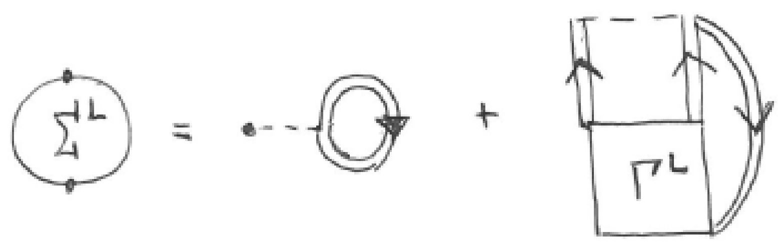
Lecture 3. Ladder approximation Single-particle propagator in a uniform system

- Self-energy and the vertex function.
- Ladder approximation to the vertex function.
- Propagation of particles and holes.
- Two-times two-body propagator.
- Analytical structure of the self-energy. Dispersion relations.
- Pairing instabilities
- Finite temperature
- Results

The single-particle propagator is defined by the irreducible self-energy through the Dyson equation



The approximation used to calculate the self-energy defines the approximation in the Green's function. The presence of a strong repulsion at short distances in the NN-interaction implies that at least one should take into account the ladder diagrams

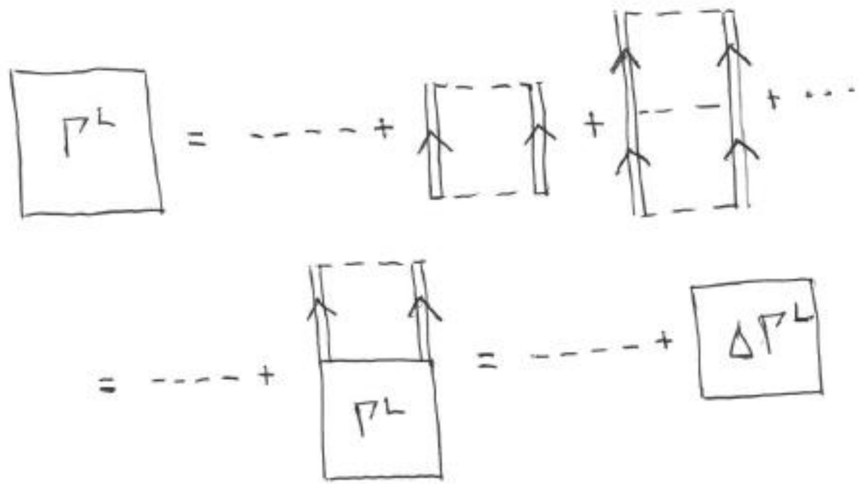


The first term is similar to the HF using the bare potential and n(k).

In this case the diagrams are considered with antisymmetrized matrix elements

The vertex function in the ladder approximation.

Notice that the intermediate two-particle states are described by dressed single-particle propagators. The intermediate states take into account the propagation of particles and holes.



The vertex function fulfils an integral equation (Lipmann-Schwinger type).

This equation has to be solved together with the Dyson equation for the single particle propagator. This is a non-linear problem which has to be solved self-consistently, following an interative procedure.
On the one hand, the interaction affects the single-particle properties but at the same time the interaction in the medium is modified by the properties of the particles in the medium

Lipmann-Schwinger equation for the vertex function in the ladder approximation

$$\langle \vec{k}\; m_\alpha\; m'_\alpha \,|\, \Gamma\,(\vec{K}_{CM}, E) \,|\, \vec{k}'\; m_\beta\; m'_\beta \rangle$$

$$= \langle \vec{k}\; m_\alpha\; m'_\alpha \,|\, V \,|\, \vec{k}'\; m_\beta\; m'_\beta \rangle \; +$$

$$+ \langle \vec{k}\; m_\alpha\; m'_\alpha \,|\, \Delta\Gamma\,(\vec{K}_{CM}, E) \,|\, \vec{k}'\; m_\beta\; m'_\beta \rangle$$

$$= \langle \vec{k}\; m_\alpha\; m'_\alpha \,|\, V \,|\, \vec{k}'\; m_\beta\; m'_\beta \rangle \; +$$

$$+ \frac{1}{2} \sum_{m_\gamma m'_\gamma} \int \frac{d^3q}{(2\pi)^3} \langle \vec{k}\; m_\alpha\; m'_\alpha \,|\, V \,|\, \vec{q}\; m_\gamma\; m'_\gamma \rangle$$

$$\cdot\, G^f\,(\vec{K}_{CM}, \vec{q}; E)\, \langle \vec{q}\; m_\gamma\; m'_\gamma \,|\, \Gamma\,(K_{CM}, E) \,|\, \vec{k}'\; m_\beta\; m'_\beta \rangle$$

$\vec{k}, \vec{k}'$ relative momenta

$\vec{K}_{CM}$ total momentum

$m_\alpha\; m'_\alpha$ generic spin and isospin quantum numbers

$G^f(K_{CM}, q, E)$ **Stands for the non-interacting dressed two-particle propagator that takes into account the propagation of two-dressed particles or holes.**

How to calculate this propagator?

It is calculated by the following convolution product where the two particles propagate with total energy E.

$$G^f(K_{CM}, q, E) = i\int_{-\infty}^{\infty} \frac{dE_1}{2\pi} g_1(\vec{q} + \frac{\vec{K}_{CM}}{2}, E_1) g_1(\frac{\vec{K}_{CM}}{2} - \vec{q}, E - E_1)$$

Where $g_1(k, E)$ **is the dressed single-particle propagator**

$$\boxed{g_1(k; E) = \int_{\varepsilon_F}^{\infty} dE' \frac{S_p(k; E')}{E - E' + i\eta} + \int_{-\infty}^{\varepsilon_F} dE' \frac{S_h(k; E')}{E - E' - i\eta}}$$

How to do this convolution product?

One uses explicitly the Lehmann representation of $g_1(k,E)$

$$= i\int \frac{dE'}{2\pi}\left[\underset{①\uparrow}{\int_{-\infty}^{\varepsilon_F} dE''\, \frac{S_h\left(\frac{\vec{k}_{cm}}{2}+\vec{q};E''\right)}{E'-E''-i\eta}} + \underset{②\downarrow}{\int_{\varepsilon_F}^{\infty} dE''\, \frac{S_p\left(\frac{\vec{k}_{cm}}{2}+\vec{q};E''\right)}{E'-E''+i\eta}}\right]$$

$$\left[\underset{③\downarrow}{\int_{-\infty}^{\varepsilon_F} dE'''\, \frac{S_h\left(\frac{\vec{k}_{cm}}{2}-\vec{q};E'''\right)}{E-E'-E'''-i\eta}} + \underset{④\uparrow}{\int_{\varepsilon_F}^{\infty} dE'''\, \frac{S_p\left(\frac{\vec{k}_{cm}}{2}-\vec{q};E'''\right)}{E-E'-E'''+i\eta}}\right]$$

We have contribution of the product of the terms ② x④ and ① x③

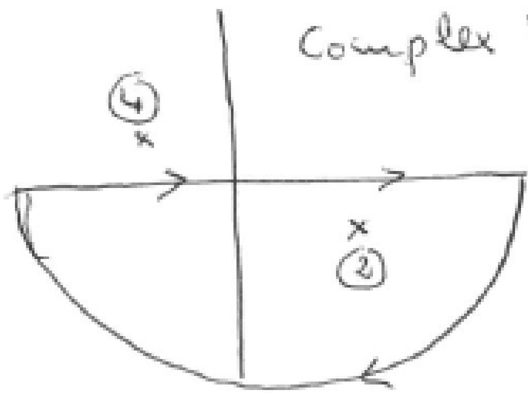
Which have the poles at different side of the complex plane E'

The other terms do not contribute because have the poles on the same side.

2 × 4

$$i\int\frac{dE'}{2\pi}\int_{\varepsilon_F}^{\infty}dE''\int_{\varepsilon_F}^{\infty}dE''' \cdot \frac{S_p\left(\frac{\vec{k}_{CM}}{2}+\vec{q}\,;E''\right)}{E'-E''+i\eta}\;\frac{S_p\left(\frac{\vec{k}_{CM}}{2}-\vec{q}\,;E'''\right)}{E-E'-E'''+i\eta}$$

complex E' plane



$$\lim_{E'\to E''-i\eta}\left(\cancel{E'-E''}+i\eta\right)\frac{S_p\left(\frac{\vec{k}_{CM}}{2}+\vec{q}\,;E''\right)}{\cancel{E'-E''}+i\eta}\;\frac{S_p\left(\frac{\vec{k}_{CM}}{2}-\vec{q}\,;E'''\right)}{E-E'-E'''+i\eta}$$

$$=\frac{S_p\left(\frac{\vec{k}_{CM}}{2}+\vec{q}\,;E''\right)S_p\left(\frac{\vec{k}_{CM}}{2}-\vec{q}\,;E'''\right)}{E-E''-E'''+i\eta}$$

$$② \times ④$$

$$i\int \frac{dE'}{2\pi}\int_{\varepsilon_F}^{\infty} dE'' \int_{\varepsilon_F}^{\infty} dE''' \; \frac{S_p\left(\frac{\vec{k}_{cM}}{2}+\vec{q};\, E''\right)}{E'-E''+i\eta} \; \frac{S_p\left(\frac{\vec{k}_{cM}}{2}-\vec{q};\, E'''\right)}{E-E'-E'''+i\eta}$$

$$= -i\,\frac{1}{2\pi}\,2\pi i \;\mathrm{Res} \;=\; \int_{\varepsilon_F}^{\infty} dE'' \int_{\varepsilon_F}^{\infty} dE''' \; \frac{S_p\left(\frac{\vec{k}_{cM}}{2}+\vec{q};\, E''\right) S_p\left(\frac{\vec{k}_{cM}}{2}-\vec{q};\, E'''\right)}{E-E''-E'''+i\eta}$$

clock-wise

The intermediate state describes the propagation of two particles or two-holes

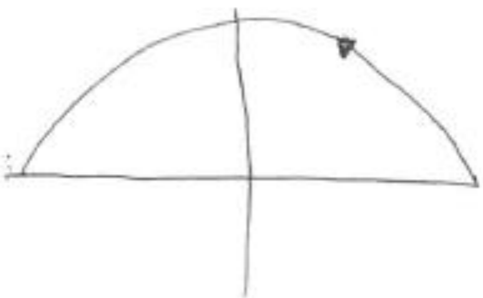
$$G^{f}(\vec{k}_{cM},\vec{q};\, E) = \int_{\varepsilon_F}^{\infty} dE'' \int_{\varepsilon_F}^{\infty} dE''' \; \frac{S_p\left(\frac{\vec{k}_{cM}}{2}+\vec{q};\, E''\right) S_p\left(\frac{\vec{k}_{cM}}{2}-\vec{q};\, E'''\right)}{E-E''-E'''+i\eta}$$

$$- \int_{-\infty}^{\varepsilon_F} dE'' \int_{-\infty}^{\varepsilon_F} dE''' \; \frac{S_h\left(\frac{\vec{k}_{cM}}{2}+\vec{q};\, E''\right) S_h\left(\frac{\vec{k}_{cM}}{2}-\vec{q};\, E'''\right)}{E-E''-E'''-i\eta}$$

The minus sign in front of the hole-hole propagtion ?

① × ③

$$i\int \frac{dE'}{2\pi}\int_{-\infty}^{\varepsilon_F} dE''\int_{-\infty}^{\varepsilon_F} dE''' \; \frac{S_h\left(\frac{\bar{k}_{CM}}{2}+\bar{q}\,;E''\right)}{E'-E''-i\eta}\;\frac{S_h\left(\frac{\bar{k}_{CM}}{2}-\bar{q}\,;E'''\right)}{E-E'-E'''-i\eta}$$



pole at E

$$\lim_{E'\to E''+i\eta}\;\cancel{(E'-E''-i\eta)}\;\frac{S_h\left(\frac{\bar{k}_{CM}}{2}+\bar{q}\,;E''\right)}{\cancel{(E'-E''-i\eta)}}\;\frac{S_h\left(\frac{\bar{k}_{CM}}{2}+\bar{q}\,;E'''\right)}{E-E'-E'''-i\eta}$$

$$=\frac{S_h\left(\frac{\bar{k}_{CM}}{2}+\bar{q}\,;E''\right)\,S_h\left(\frac{\bar{k}_{CM}}{2}-\bar{q}\,;E'''\right)}{E-E''-E'''-i\eta}$$

$$= i\,\frac{1}{2\pi}\,2\pi i\;\mathrm{Res} = -\int_{-\infty}^{\varepsilon_F} dE''\int_{-\infty}^{\varepsilon_F} dE'''\;\frac{S_h\left(\frac{\bar{k}_{CM}}{2}+\bar{q}\,;E''\right)S_h\left(\frac{\bar{k}_{CM}}{2}-\bar{q}\,;E''\right)}{E-E''-E'''-i\eta}$$

How it looks with mean field like propagators?

In this case, the spectral functions are delta functions of strength 1 located at the quasi-particle energies:

$$S_p\left(\frac{\vec{k}_{CM}}{2}+\vec{q};E''\right)=\delta\left(E''-\varepsilon\left(\left|\frac{\vec{k}_{CM}}{2}+\vec{q}\right|\right)\right)\theta\left(\left|\frac{\vec{k}_{CM}}{2}+\vec{q}\right|-k_F\right)$$

$$S_h\left(\frac{\vec{k}_{CM}}{2}+\vec{q};E\right)=\delta\left(E-\varepsilon\left(\left|\frac{\vec{k}_{CM}}{2}+\vec{q}\right|\right)\right)\theta\left(k_F-\left|\frac{\vec{k}_{CM}}{2}+\vec{q}\right|\right)$$

where $\varepsilon(|\vec{k}|)$ is the quasi-particle energy

$$\varepsilon(|\vec{k}|)=\frac{\hbar^2k^2}{2m} \quad \text{free case}$$

$$\varepsilon(|\vec{k}|)=\frac{\hbar^2k^2}{2m}+\mathrm{Re}\,\Sigma_1(k,\varepsilon(|k|))$$

→ BHF

→ Propagating particles and holes with a mean field type propagators

The convolution product in this case:

$$G^f(\vec{k}_{CM},\vec{q};E) = \int_{\varepsilon_F}^{\infty} dE'' \int_{\varepsilon_F}^{\infty} dE''' \, \frac{\delta\left(E''-\varepsilon\left(\left|\frac{\vec{k}_{CM}}{2}+\vec{q}\right|\right)\right)\theta\left(\left|\frac{\vec{k}_{CM}}{2}+\vec{q}\right|-k_F\right)\delta\left(E'''-\varepsilon\left(\left|\frac{\vec{k}_{CM}}{2}-\vec{q}\right|\right)\right)\theta\left(\left|\frac{\vec{k}_{CM}}{2}-\vec{q}\right|-k_F\right)}{E-E''-E'''+i\eta}$$

$$- \int_{-\infty}^{\varepsilon_F} dE'' \int_{-\infty}^{\varepsilon_F} dE''' \, \frac{\delta\left(E''-\varepsilon\left(\left|\frac{\vec{k}_{CM}}{2}+\vec{q}\right|\right)\right)\theta\left(k_F-\left|\frac{\vec{k}_{CM}}{2}+\vec{q}\right|\right)\cdot\delta\left(E'''-\varepsilon\left(\left|\frac{\vec{k}_{CM}}{2}-\vec{q}\right|\right)\right)\theta\left(k_F-\left|\frac{\vec{k}_{CM}}{2}-\vec{q}\right|\right)}{E-E''-E'''-i\eta}$$

$$=$$

The deltas allow to perform the integrals

$$G^f(\vec{k}_{cm},\vec{q};E)=\frac{\theta\left(\left|\frac{\vec{k}_{cm}}{2}+\vec{q}\right|-k_F\right)\theta\left(\left|\frac{\vec{k}_{cm}}{2}-\vec{q}\right|-k_F\right)}{E-\varepsilon\left(\left|\frac{\vec{k}_{cm}}{2}+\vec{q}\right|\right)-\varepsilon\left(\left|\frac{\vec{k}_{cm}}{2}-\vec{q}\right|\right)}$$

$$-\frac{\theta\left(k_F-\left|\frac{\vec{k}_{cm}}{2}+\vec{q}\right|\right)\quad\theta\left(k_F-\left|\frac{\vec{k}_{cm}}{2}-\vec{q}\right|\right)}{E-\varepsilon\left(\left|\frac{\vec{k}_{cm}}{2}+\vec{q}\right|\right)-\varepsilon\left(\left|\frac{\vec{k}_{cm}}{2}-\vec{q}\right|\right)}$$

Which coincides with :

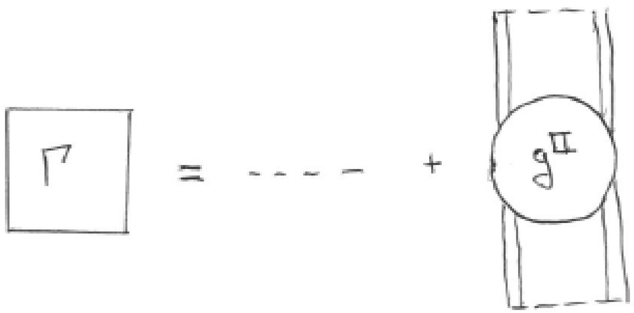
$$G^f(\vec{k}_{cm},\vec{q};E)=i\int\frac{dE'}{2\pi}\,g^{(0)}\left(\frac{\vec{k}_{cm}}{2}+\vec{q};E'\right)g^{(0)}\left(\frac{\vec{k}_{cm}}{2}-\vec{q};E-E'\right)$$

$$g^{(0)}(k,E)=\frac{\theta(k-k_F)}{E-\varepsilon(k)-i\eta}+\frac{\theta(k_F-k)}{E-\varepsilon(k)+i\eta}$$

The vertex function in the ladder approximation can be directly related to the two-times two-particle propagator

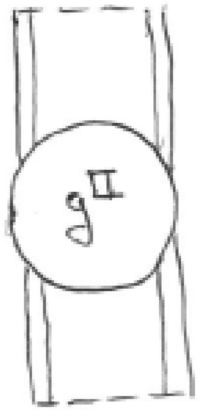
$$g^{II}(\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4; t_2-t_1) = -\frac{i}{\hbar}\,\frac{\langle\Psi_0|\,T\{a_{\bar{k}_1}(t_1)\,a_{\bar{k}_2}(t_1)\,a^{+}_{\bar{k}_3}(t_2)\,a^{+}_{\bar{k}_4}(t_2)\}\,|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}$$

With the following schematic diagrammatic equation:



Inserting a complete set of states of the A-2 and the A+2 system and perfor ming a Fourier transform one gets the corresponding Lehmann representation

$$g^{II}(\bar{q},\bar{q}';\vec{k}_{CM};E) = \int_{2\varepsilon_F}^{\infty} dE'\,\frac{S_{pp}(\bar{q},\bar{q}',\bar{k}_{CM};E')}{E-E'+i\eta} - \int_{-\infty}^{2\varepsilon_F} dE'\,\frac{S_{hh}(\bar{q},\bar{q}',\bar{k}_{CM};E')}{E-E'-i\eta}$$



$$\langle \vec{k}_r \,|\, \Delta\Gamma(\vec{k}_{CM}, E) \,|\, \vec{k}'_r \rangle =$$

$$= \frac{1}{2}\int \frac{d^3q}{(2\pi)^3} \, \frac{1}{2}\int \frac{d^3q'}{(2\pi)^3} \langle \vec{k}_r | V | \vec{q} \rangle \, g^{II}(\vec{q}, \vec{q}\,'; \vec{K}_{CM}; E) \langle \vec{q}\,' | V | \vec{k}'_r \rangle$$

Introducing the Lehmann representation

$$= \frac{1}{2}\int \frac{d^3q}{(2\pi)^3} \, \frac{1}{2}\int \frac{d^3q'}{(2\pi)^3} \langle \vec{k}_r | V | \vec{q} \rangle$$

$$\left\{ \int_{2\varepsilon_F}^{\infty} dE' \, \frac{S_{pp}(\vec{q}, \vec{q}\,'; \vec{K}_{CM}; E')}{E - E' + i\eta} - \int_{-\infty}^{2\varepsilon_F} dE' \, \frac{S_{hh}(\vec{q}, \vec{q}\,'; \vec{K}_{CM}; E')}{E - E' - i\eta} \right\} \langle \vec{q}\,' | V | \vec{k}'_r \rangle$$

To find the imaginary part of the matrix gamma, we take into account that

$$Im\Gamma(\vec{k}_r, \vec{k}'_r, \vec{K}_{CM}, E) = Im\Delta\Gamma(\vec{k}_r, \vec{k}'_r, \vec{K}_{CM}, E)$$

The imaginary part of gamma,

$$\mathrm{Im}\,\Gamma(\vec{k}_r, \vec{k}_r', \vec{k}_{cm}; E) =$$

$$-\pi \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\,\frac{1}{2}\int\frac{d^3q'}{(2\pi)^3}\langle\vec{k}_r|V|\vec{q}\rangle\, S_{pp}(\vec{q},\vec{q}';\vec{k}_{cm};E)\langle\vec{q}'|V|\vec{k}_r'\rangle \qquad E > 2\varepsilon_F$$

$$-\pi \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\,\frac{1}{2}\int\frac{d^3q'}{(2\pi)^3}\langle\vec{k}_r|V|\vec{q}\rangle\, S_{hh}(\vec{q},\vec{q}';\vec{k}_{cm};E)\langle\vec{q}'|V|\vec{k}_r'\rangle \qquad E < 2\varepsilon_F$$

A dispersion relation:

$$\Delta\Gamma(\vec{k}_r, \vec{k}_r', \vec{k}_{cm}, E) =$$

$$-\frac{1}{\pi}\int_{2\varepsilon_F}^{\infty} dE'\,\frac{\mathrm{Im}\,\Gamma(\vec{k}_r, \vec{k}_r', \vec{k}_{cm}; E')}{E - E' + i\eta}$$

$$+\frac{1}{\pi}\int_{-\infty}^{2\varepsilon_F} dE'\,\frac{\mathrm{Im}\,\Gamma(\vec{k}_r, \vec{k}_r', \vec{k}_{cm}; E')}{E - E' - i\eta}$$

Can be separated into two parts each one having singularities in one half of the complex energy plane.

$$\Delta\Gamma(\vec{k}_r, \vec{k}_r', \vec{k}_{CM}, E) \equiv \Delta^{\downarrow}\Gamma(\vec{k}_r, \vec{k}'_r, \vec{k}_{CM}; E) + \Delta^{\uparrow}\Gamma(\vec{k}_r, \vec{k}'_r, \vec{k}_{CM}; E)$$

These two pieces are obtained from a dispersion integral over the imaginary part of Gamma. The pp part, with poles in the lower half plane, results from integrating Im (Gamma) along the energies available to the (A+2) states, Whereas the hh part presents singularities in the upper half plane and the integration runs along the excitation energies of the (A-2) system.

$$\boxed{\text{The knowledge of the imaginary part of the effective interaction Gamma and the bare potential allows to recover the full (real and imaginary) effective interaction.}}$$

This separation of Gamma is crucial for the correct evaluation of the self-energy.

$$\Sigma_1(k,E) = \int \frac{d^3k'}{(2\pi)^3} \oint \frac{dE'}{2\pi i} \langle \vec{k}_r | V | \vec{k}_r \rangle \, g(k',E')$$

$$+ \int \frac{d^3k'}{(2\pi)^3} \oint \frac{dE'}{2\pi i} \langle \vec{k}_r | \Delta\Gamma(\vec{K}_{CM}, E+E') | \vec{k}_r \rangle \, g(k',E')$$

Remember diagrammatic rules!

The evaluation of the first term is straigthforward since the V does not depend on the energy

$$\Sigma_1^V(k) = \int \frac{d^3k'}{(2\pi)^3} \int_{-\infty}^{\varepsilon_F} dE' \langle \vec{k}_r | V | \vec{k}_r \rangle \, S_h(k',E')$$

$$= \int \frac{d^3k'}{(2\pi)^3} \langle \vec{k}_r | V | \vec{k}_r \rangle \, n(k')$$

In HF or BHF

$$\boxed{n(k') = \Theta(k_F - k')}$$

To evaluate the second piece one uses the decompostion of the Gamma matrix and the Lehmann representation of the single-particle propagator

$$\Sigma_{\Delta\Gamma}(\vec{k}, E) = \int \frac{d^3k'}{(2\pi)^3} \int \frac{dE'}{2\pi i} \Delta\Gamma(\vec{k}_r, \vec{k}_r, \vec{k}_{CM}; E+E')\, g(k', E')$$

$$= \int \frac{d^3k'}{(2\pi)^3} \int \frac{dE'}{2\pi i} \left\{ -\frac{1}{\pi} \int_{2\varepsilon_F}^{\infty} dE'' \frac{\mathrm{Im}\, \Delta\Gamma(\vec{k}_r, \vec{k}_r, \vec{k}_{CM}; E'')}{E+E'-E''+i\eta} + \right.$$

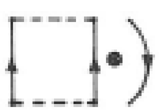
$$\left. + \frac{1}{\pi} \int_{-\infty}^{2\varepsilon_F} dE'' \frac{\mathrm{Im}\, \Delta\Gamma(\vec{k}_r, \vec{k}_r, \vec{k}_{CM}; E'')}{E+E'-E''-i\eta} \right\}$$

$$\left\{ \int_{\varepsilon_F}^{\infty} dE''' \frac{S_p(k', E''')}{E'-E'''+i\eta} + \int_{-\infty}^{\varepsilon_F} dE''' \frac{S_h(k', E''')}{E'-E'''-i\eta} \right\}$$

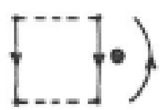
The pp-Gamma is closed by the hole part of g(k,E) and the hh-Gamma by the particle part.

$$\Sigma'_{\Delta\Gamma}(k,E) = \int\frac{d^3k'}{(2\pi)^3}\int_{-\infty}^{\varepsilon_F} dE''' \left\{-\frac{1}{\pi}\int_{2\varepsilon_F}^{\infty} dE'' \frac{\mathrm{Im}\,\Delta\Gamma(k_r,k_r,k_{CM},E'')}{E+E'''-E''+i\eta}\right\} S_h(k',E''')$$

$$- \int\frac{d^3k'}{(2\pi)^3}\int_{\varepsilon_F}^{\infty} dE'''\left\{\frac{1}{\pi}\int_{-\infty}^{2\varepsilon_F} dE''\,\frac{\mathrm{Im}\,\Delta\Gamma(k_r,k_r,k_{CM};E'')}{E+E'''-E''-i\eta}\right\} S_p(k',E''')$$



a



b

Diagram (a) has poles in the lower-half, belongs to $\Delta^{\downarrow}\Gamma$

It should be closed by the hole part of g(k,E)

Diagram (b) has poles in the upper-part and should be closed by the particle part of g(k,E).

It belongs to $\Delta^{\uparrow}\Gamma$

Allows to decompose the self-energy in two pieces according to its analytical structure

$$\Sigma'_{\Delta\Gamma}(\bar{k},E) = \int \frac{d^3k'}{(2\pi)^3} \int_{-\infty}^{\varepsilon_F} dE''' \, \Delta^{\downarrow}\Gamma(\bar{k}_r, \bar{k}_r', \bar{k}_{CM}, E+E''') \, S_h(k', E''')$$

$$- \int \frac{d^3k'}{(2\pi)^3} \int_{\varepsilon_F}^{\infty} dE''' \, \Delta^{\uparrow}\Gamma(\bar{k}_r, \bar{k}_r', \bar{k}_{CM}, E+E''') \, S_p(k', E''')$$

$$\equiv \Delta\Sigma'_{\downarrow}(k,E) + \Delta\Sigma'_{\uparrow}(k,E)$$

Therefore the imaginary part of $\Sigma^{\downarrow}_{\Delta\Gamma}(\bar{k}, E)$.

$$\boxed{\mathrm{Im}\,\Sigma^{\downarrow}_{\Delta\Gamma}(\bar{k}, E) = \int \frac{d^3k'}{(2\pi)^3} \int_{-\infty}^{\varepsilon_F} dE''' \,\mathrm{Im}\,\Delta\Gamma(\bar{k}_r, \bar{k}_r, \bar{k}_{CM}, E+E''')\, S_h(k', E''')}$$

2p1h

And the same for $\Sigma^{\uparrow}_{\Delta\Gamma}(\bar{k}, E)$

$$\boxed{\mathrm{Im}\,\Sigma^{\uparrow}_{\Delta\Gamma}(k, E) = -\int \frac{d^3k'}{(2\pi)^3} \int_{\varepsilon_F}^{\infty} dE''' \,\mathrm{Im}\,\Delta\Gamma(\bar{k}_r, \bar{k}_r, \bar{k}_{CM}, E+E''')\, S_p(k', E''')}$$

2h1p

Finally, the self-energy can be written as a real energy independent part and a complex contribution which can be calculated from the imaginary part through a dispersion relation

$$\Sigma_1(k,E) = \Sigma_{1V}(k) - \frac{1}{\pi}\int_{\varepsilon_F}^{\infty} dE' \, \frac{\mathrm{Im}\,\Sigma_1(k,E')}{E-E'+i\eta} + \frac{1}{\pi}\int_{-\infty}^{\varepsilon_F} dE' \, \frac{\mathrm{Im}\,\Sigma_1(k,E')}{E-E'-i\eta}$$

$$= \Sigma_V(k) + \Delta\Sigma_{1f}(k,E) + \Delta\Sigma_{1p}(k,E)$$

Once the self-energy is known one has access to g(k,E) through the Dyson equation and to the spectral functions as the imaginary part of g(k,E).

$$S_p(k,E) = -\frac{1}{\pi} \frac{\mathrm{Im}\,\Sigma^1_4(k,E)}{\left(E - \frac{\hbar^2 k^2}{2m} - \mathrm{Re}\,\Sigma^1_4(k,E)\right)^2 + \left(\mathrm{Im}\,\Sigma^1_4(k,E)\right)^2}$$

$$S_h(k,E) = \frac{1}{\pi} \frac{\mathrm{Im}\,\Sigma^1_4(k,E)}{\left(E - \frac{\hbar^2 k^2}{2m} - \mathrm{Re}\,\Sigma^1_4(k,E)\right)^2 + \left(\mathrm{Im}\,\Sigma^1_4(k,E)\right)^2}$$

The momentum dsitribution

$$n(k) = \int_{-\infty}^{\epsilon_F} dE S_h(k,E)$$

The energy per particle

$$\frac{B}{A} = \frac{\nu}{\rho} \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\epsilon_F} dE\, S_h(k,E) \left(\frac{\hbar^2 k^2}{2m} + E\right)$$

From the signs of the imaginary part of the self-energy we can, through the dispersion relation, see if the real part will be attractive or repulsive!

$$Im\,\Sigma_{\uparrow}'(k,E) = Im\,\Delta\Sigma_{\uparrow}'(k,E) > 0 \qquad E < \varepsilon_F$$

$$Im\,\Sigma_{\downarrow}(k,E) = Im\,\Delta\Sigma_{\downarrow}'(k,E) < 0 \qquad E > \varepsilon_F$$

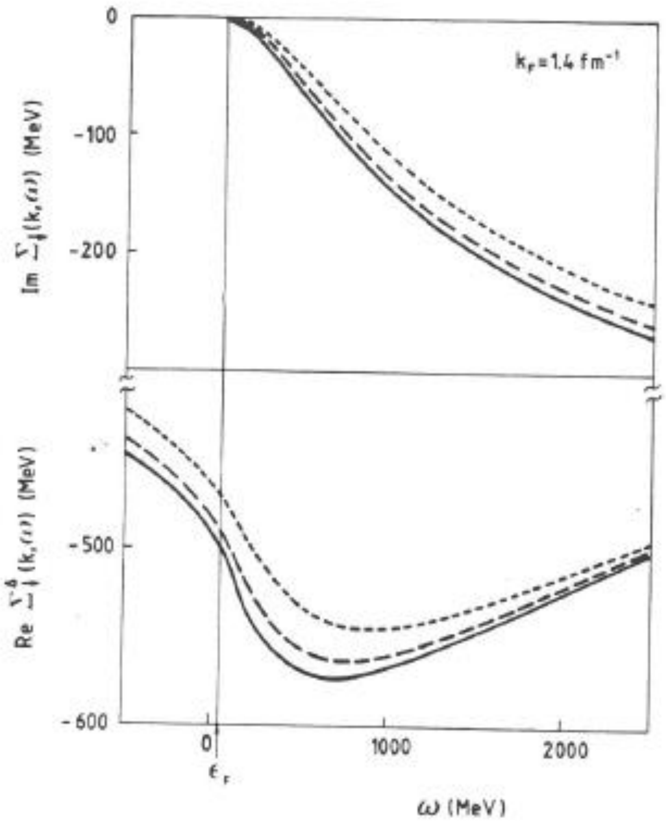
$$\boxed{Re\Sigma_{\downarrow}^{\Delta}(k,E) = -\frac{P}{\pi}\int_{\epsilon_F}^{\infty} dE' \frac{Im\Sigma_{\downarrow}^{\Delta}(k,E')}{E-E'}} \qquad Re\Delta\Sigma_{\downarrow}(k,E) < 0, E < \epsilon_F$$

For energies E below the Fermi energy that will be an attractive contribution

$$\boxed{\Sigma_{\uparrow}^{\Delta}(k,E) = \frac{P}{\pi}\int_{-\infty}^{\epsilon_F} dE' \frac{Im\Sigma_{\uparrow}^{\Delta}(k,E')}{E-E'}} \qquad Re\Delta\Sigma_{\uparrow}(k,E) > 0, E > \epsilon_F$$

Which for energies above the Fermi energy will give a repulsive contribution

Due to the dispersion relation and to the well define sign o the imaginary part, one can imagine the general shape of the real part of the self-energy and predict if the contributions will be attractive or repulsive



Energy dependence of the imaginary (upper part) and real (lower part) components of

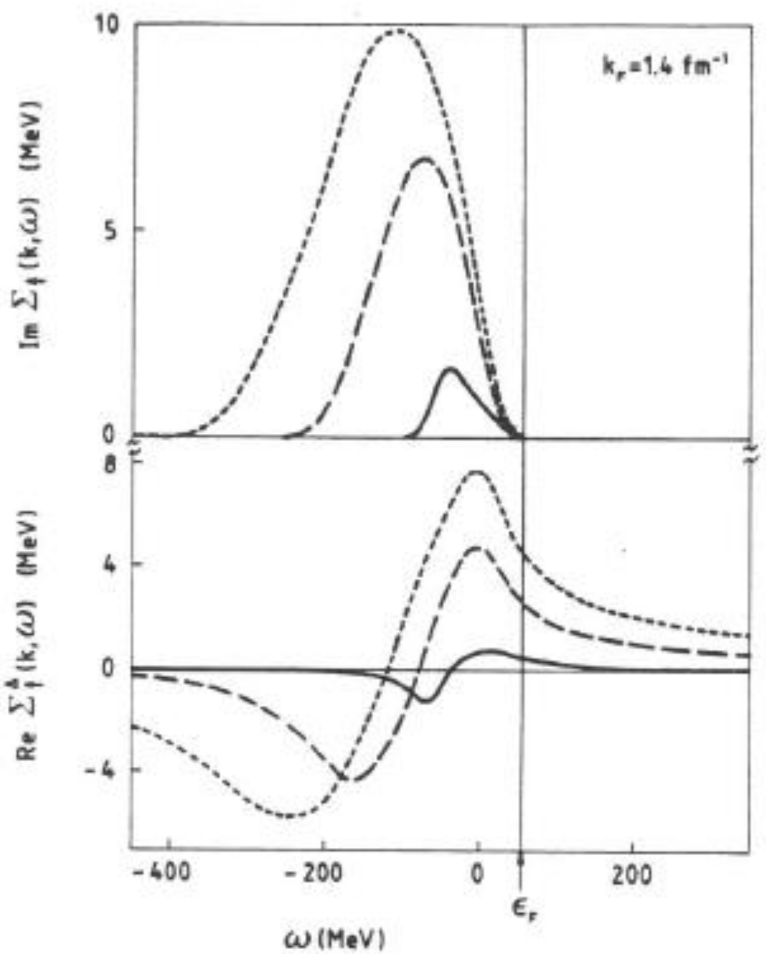
$$\Sigma_{\downarrow}^{\Delta}(k, E)$$

For three different momenta

$$k = 0.1\ \mathrm{k}_F,\ k = 0.9\ \mathrm{k}_F,\ k = 1.6\ \mathrm{k}_F$$

Does not correspond to a realistic Potential.

Similar analysis for $\Sigma_{\uparrow}^{\Delta}(k, E)$



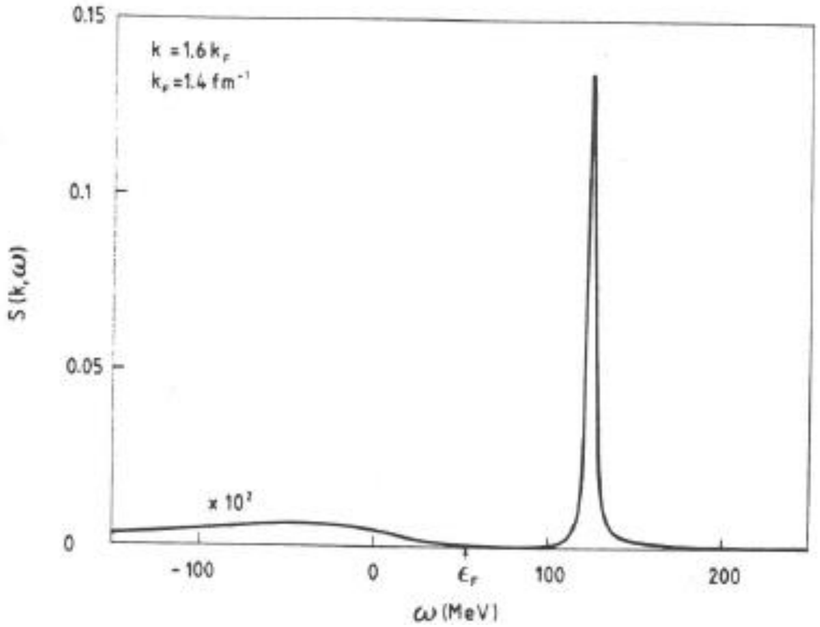
Energy dependence of the imaginary (upper part) and real (lower part) Components of the self-energy.

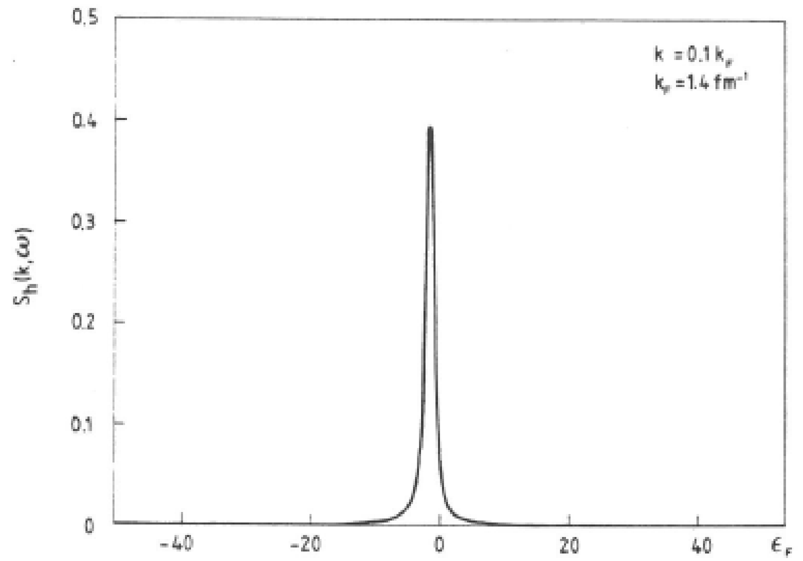
$k = 0.1\ \mathrm{k}_F$ (Solid line)

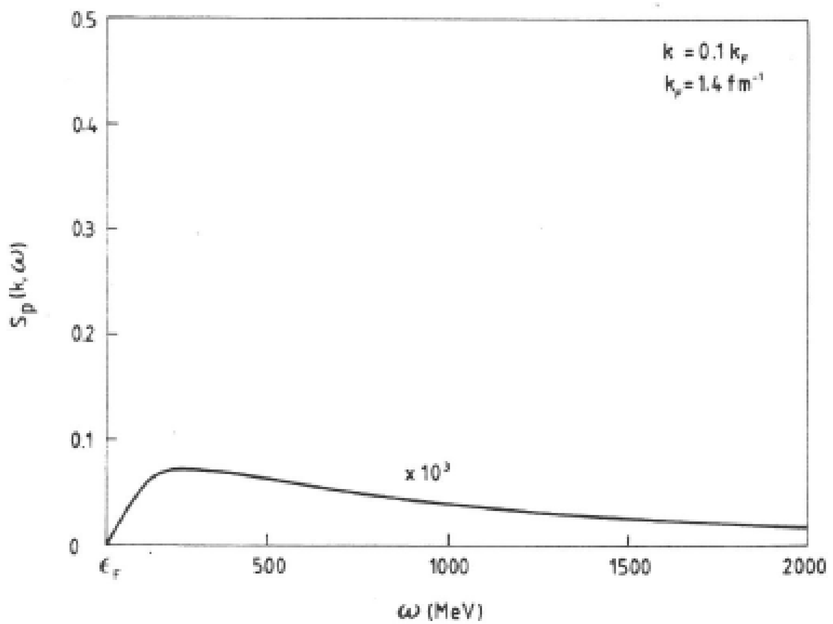
$k = 0.9\ \mathrm{k}_F$ (long-dashed line)

$k = 1.6\ \mathrm{k}_F$ (dashed line)

Characteristic shapes of Spectral functions







Can we recover the BHF approximation from the SCGF?

One recovers the G-matrix by considering only two-particle states as intermediate states, dressed with a mean field propagator built with the real part of the self-energy on shell

$$< \mathbf{k_r} \mid G(\mathbf{K},\Omega) \mid \mathbf{k'_r} > = < \mathbf{k_r} \mid V \mid \mathbf{k'_r} > +$$

$$+ \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} < \mathbf{k_r} \mid V \mid \mathbf{q} > \frac{\theta(\mid \frac{\vec{K}}{2} + \mathbf{q} \mid - k_F)\theta(\mid \frac{\vec{K}}{2} - \mathbf{q} \mid - k_F)}{\Omega - \epsilon(\left|\frac{\vec{K}}{2} + \mathbf{q}\right|) - \epsilon(\left|\frac{\vec{K}}{2} - \mathbf{q}\right|) + i\eta} < \mathbf{q} \mid G(\mathbf{K},\Omega) \mid \mathbf{k'_r} >$$

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m} + U(k) \qquad U(k) = Re \sum_{j<k_F} < k\, j \mid G(\epsilon(k) + \epsilon(j)) \mid k\, j > \qquad \forall \;\; k$$

$$S_h(k,E) = \delta(E - \epsilon_B(k))\theta(k_F - k)$$

$$\frac{B}{A} = \frac{\nu}{\rho} \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\varepsilon_F} dE \left(\frac{\hbar^2 k^2}{2m} + E\right) \delta(E - \varepsilon_B(k))\, \theta(k_F - k)$$

$$= \frac{\nu}{\rho} \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left(\frac{\hbar^2 k^2}{2m} + \varepsilon_B(k)\right) \theta(k_F - k)$$

Energy per particle, from the Koltun sum-rule

$$\frac{B}{A} = \frac{\nu}{\rho}\,\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\varepsilon_F} dE\left(\frac{\hbar^2k^2}{2m}+E\right)\delta\left(E-\varepsilon_B(k)\right)\Theta(k_F-k)$$

$$= \frac{\nu}{\rho}\,\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\left(\frac{\hbar^2k^2}{2m}+\varepsilon_B(k)\right)\Theta(k_F-k)$$

Which finally results in the kinetic energy and the average of the Single-particle potential, with a factor 1/2 to avoid double counting

$$= \frac{\nu}{\rho}\,\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,2\,\frac{\hbar^2k^2}{2m} + \frac{\nu}{\rho}\,\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,U_{BHF}(k)$$

Attention! All these calculations propagating particles and holes can show pairing instabilities when the interaction is attractive. In fact, one can have bound states of two-particles respect to the continuum of the two particles states, i.e. below twice the Fermi energy. These instabilities appear as a pole in the complex plane. The proper treatment of these bound states requires an extension of the SCGF formalism to take into account the short-range correlations together with the long-range correlations associated to the pairing phenomena.

We have the pole of the deuteron in the T-matrix in free space when we study the 3S1-3D1 cannel. Then for small densities this bound state, still exists when one increases the density. Of course the shape and the binding energy of this bound state changes with the density.

In the G-matrix, as we do not explore the energy regions where the bound states can be located, we are not so sensitive to the existence of these bound states.

A possible strategy! Increase the temperature in such a way that the bound state dissapears! At T= 5 MeV , the pairing problem dissapears! One can then try to extrapolate the energy results to T=0, and to determine the equation of state at zero temperature. For neutron matter, where the 3S1-3D1 cannel is not active the problem is more suitable. However, there are other in channels in which the pairing problem shows up.

Single particle propagator

Zero temperature

$$ig(\mathbf{k}t, \mathbf{k}'t') = \langle \Psi_0 \mid T\left[a_{\mathbf{k}}(t)a^\dagger_{\mathbf{k}'}(t')\right] \mid \Psi_0\rangle$$

Heisenberg picture $$a_{\mathbf{k}}(t) = e^{it\hat{H}} a_{\mathbf{k}} e^{-it\hat{H}}$$

T is the time ordering operator

Finite temperature

$$ig(\mathbf{k}t, \mathbf{k}'t') = Tr\left\{\rho T\left[a_{\mathbf{k}}(t)a^\dagger_{\mathbf{k}'}(t')\right]\right\}$$

The trace is to be taken over all energy eigenstates and all particle number eigenstates of the many-body system

$$\rho = \frac{1}{Z} e^{-\beta(\hat{H}-\mu\hat{N})}$$

$$Z = Tre^{-\beta(\hat{H}-\mu\hat{N})}$$

✓**Z is the grand partition function**

Lehmann representation + Spectral functions

FT+ clossure => Lehmann representation

$$g(k,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \frac{A^{>}(k,\omega')}{\omega-\omega'+i\eta} + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \frac{A^{<}(k,\omega')}{\omega-\omega'-i\eta}$$

$$A^{<}(k,\omega) = 2\pi \sum_{n,m} \frac{e^{-\beta(E_m-\mu N_m)}}{Z} \mid \langle \Psi_n \mid a_{\mathbf{k}} \mid \Psi_m \rangle \mid^2 \delta(\omega-(E_m-E_n))$$

$$A^{>}(k,\omega) = 2\pi \sum_{m,n} \frac{e^{-\beta(E_m-\mu N_m)}}{Z} \mid \langle \Psi_n \mid a_{\mathbf{k}}^{\dagger} \mid \Psi_m \rangle \mid^2 \delta(\omega-(E_n-E_m))$$

The summation runs over all energy eigenstates and all particle number eigenstates

The spectral function

$$\boxed{A(k,\omega) = A^{<}(k,\omega) + A^{>}(k,\omega)}$$

with $\boxed{A^{>}(k,\omega) = e^{\beta(\omega-\mu)} A^{<}(k,\omega)}$ therefore $\boxed{A^{<}(k,\omega) = A(k,\omega) f(\omega)}$

where $\boxed{f(\omega) = \left\{ e^{\beta(\omega-\mu)} + 1 \right\}^{-1}}$ Is the Fermi function

and $A^{>}(k,\omega) = A(k,\omega)(1 - f(\omega))$

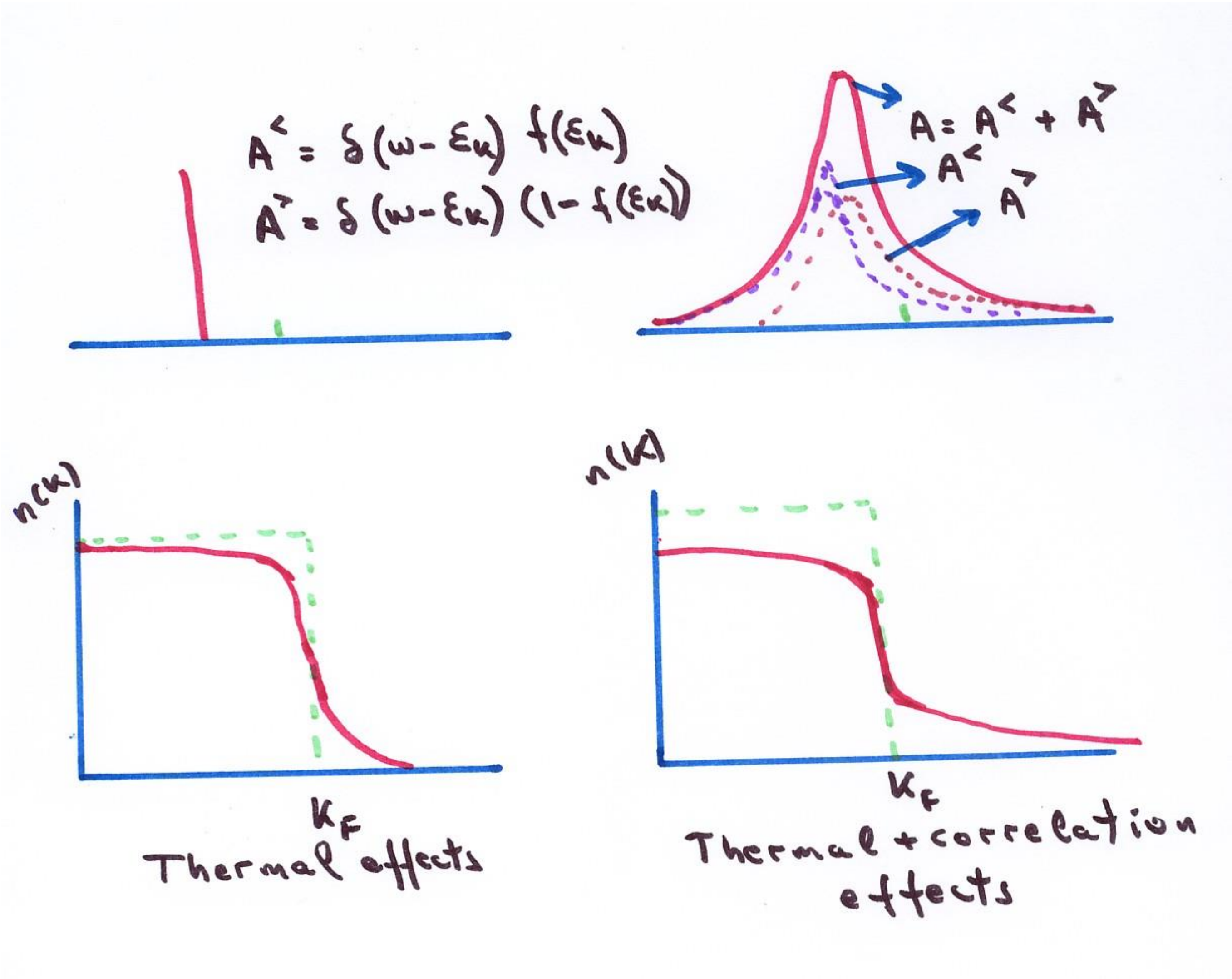
Momentum distribution

T=0 MeV $$n(k, T=0) = \frac{1}{2\pi} \int_{-\infty}^{\epsilon_F} A_h(k,\omega) d\omega$$

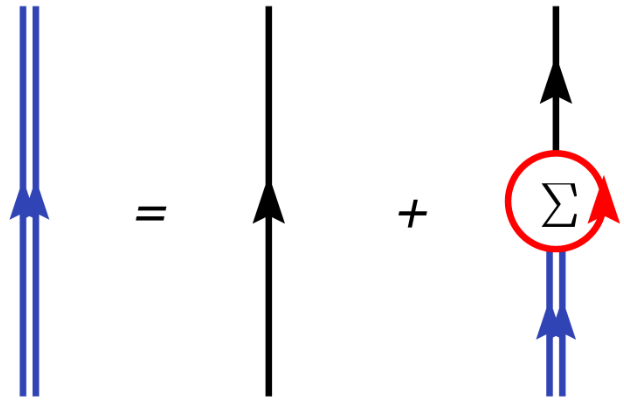
Finite T $$\boxed{n(k,T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A^{<}(k,\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(k,\omega) f(\omega) d\omega}$$

Spectral functions at finite Temperature

Free system + Interactions => Correlated system



Dyson equation



$$A(k,\omega) = \frac{-2Im\Sigma(k,\omega+i\eta)}{\left[\omega - \frac{k^2}{2m} - Re\Sigma(k,\omega)\right]^2 + \left[Im\Sigma(k,\omega+i\eta)\right]^2}$$

$$g(k,\omega) = \int_{-\infty}^{\infty} d\omega' \left[\frac{A(k,\omega')f(\omega')}{\omega - \omega' - i\eta} + \frac{A(k,\omega')(1-f(\omega'))}{\omega - \omega' + i\eta} \right]$$

From the density normalization condition, one gets a microscopic determination of the chemical potential

$$\boxed{\rho = \nu \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathcal{A}(k,\omega) f(\omega,\tilde{\mu})}$$

The Koltun sum-rule is also valid at finite T. One should include the Fermi factor.

$$\boxed{\frac{E}{A} = \frac{\nu}{\rho} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{1}{2} \left\{ \frac{k^2}{2m} + \omega \right\} \mathcal{A}(k,\omega) f(\omega)}$$

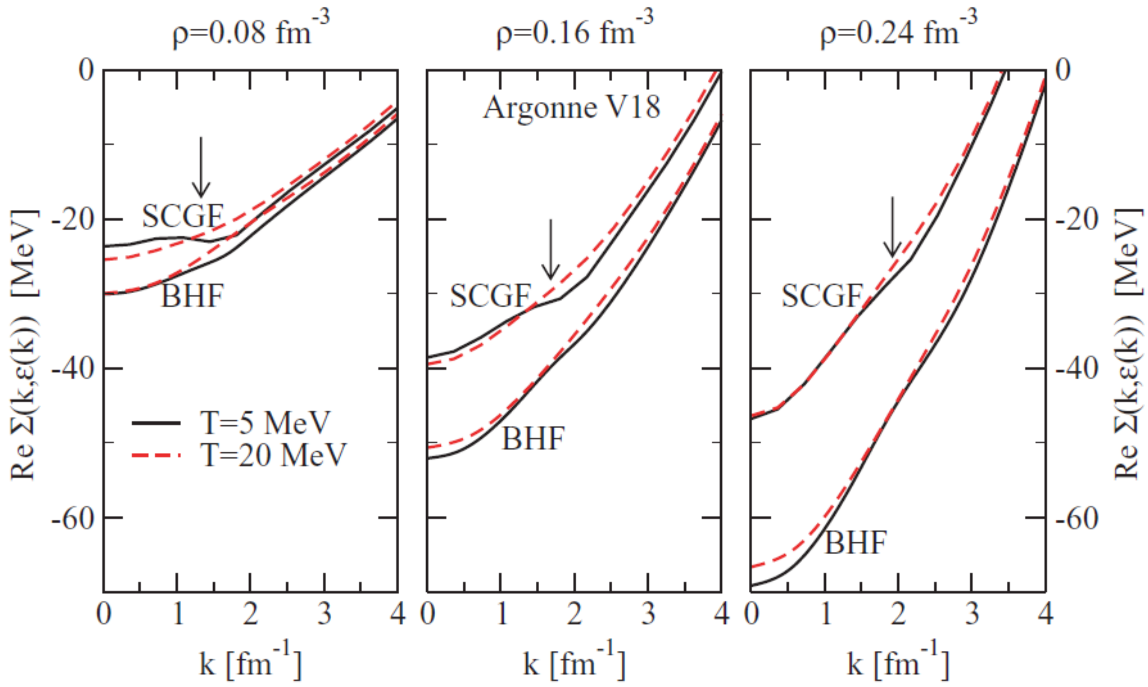
The BHF approach is obtained by propagating only particles in the intermediate states and assuming that the spectral function has no width and all its strength is concentrated in the BHF quasiparticle energy

$$\mathcal{A}(k,\omega) = (2\pi)\delta\left[\omega - \varepsilon_{\mathrm{BHF}}(k)\right]$$

$$\mathcal{G}^0_{\mathrm{II}}(k,k';\Omega_+) = \frac{[1-f(\omega)][1-f(\omega')]}{\Omega_+ - \varepsilon_{\mathrm{BHF}}(k) - \varepsilon_{\mathrm{BHF}}(k')}$$

The Pauli operator is generalized using the thermal momentum distributions instead of the step functions.

Attention! The BHF approach is not thermodynamically consistent, in the sense that the chemical potential extracted from the density normalization condition differs considerably from the one calculated from the derivative of the free-energy.



Neutron matter

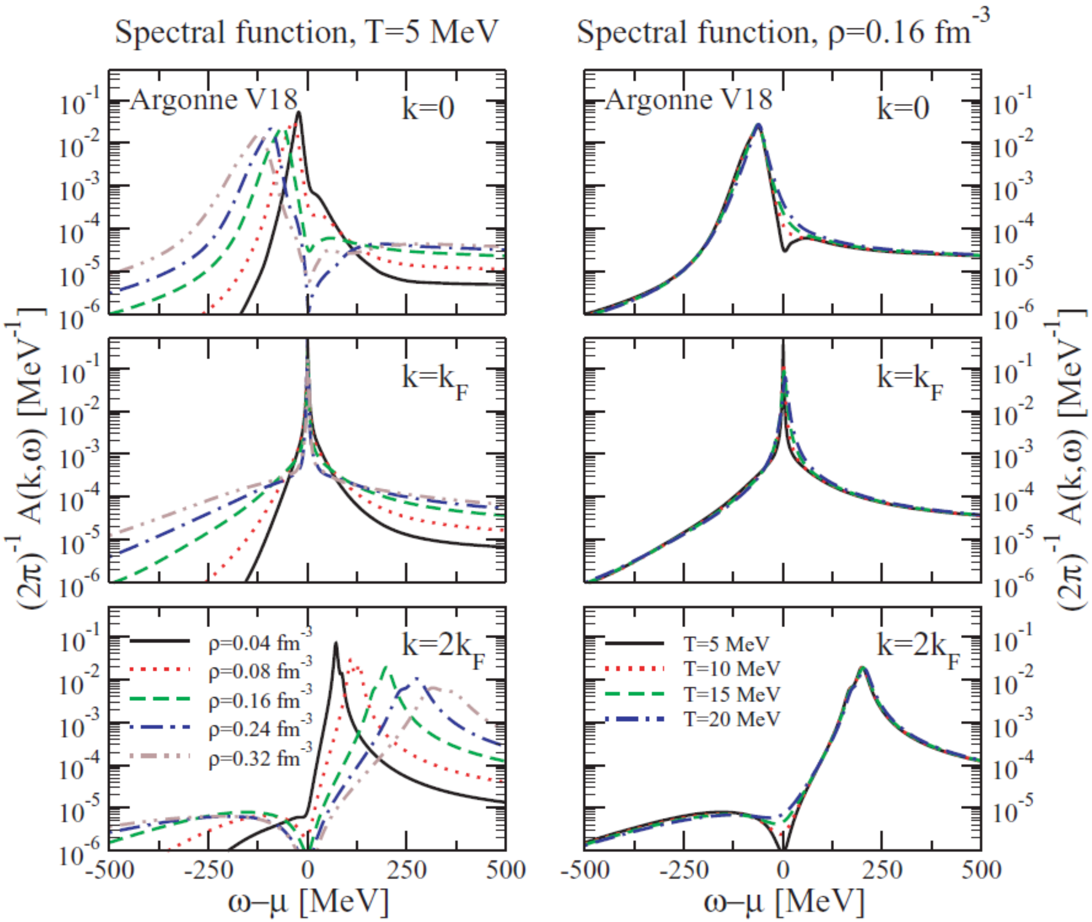
Thermal effects are small

Real part of the on-Shell self-energy as a function of k for the SCGF and BHF approximations at T=5 MeV (solid lines) and T=20 MeV (dashed lines). For three different densities.

For all densities the SCGF spectra are more repulsive tan the BHF ones at all momenta. PROPAGATION OF HOLES PRODUCE REPULSION

The effect is larger for low momenta. The repulsive effect increases with density

The repulsive effect is around 25 MeV for k=0 at $\rho = 0.24\ \mathrm{fm}^{-3}$

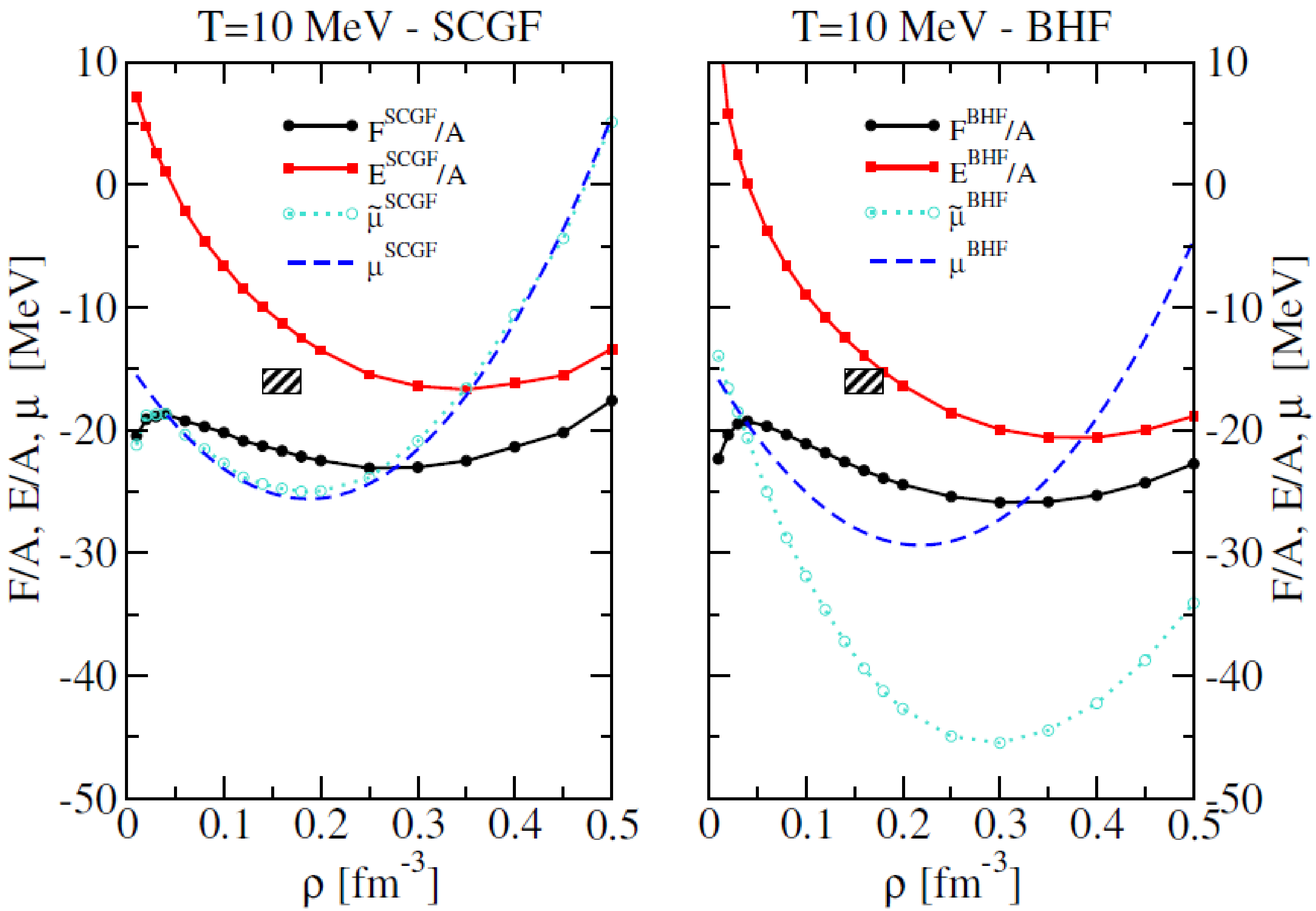


Density (left panels) and temperature (right panels) dependence of the spectral function as a function of the energy for three momenta.

There is an important quasiparticle peak, which contains around 70%-80% of the total strength for all momenta , located at

$$\varepsilon_{qp}(k) = \frac{\hbar^2 k^2}{2m} + \mathrm{Re}\Sigma[k, \varepsilon_{qp}(k)]$$

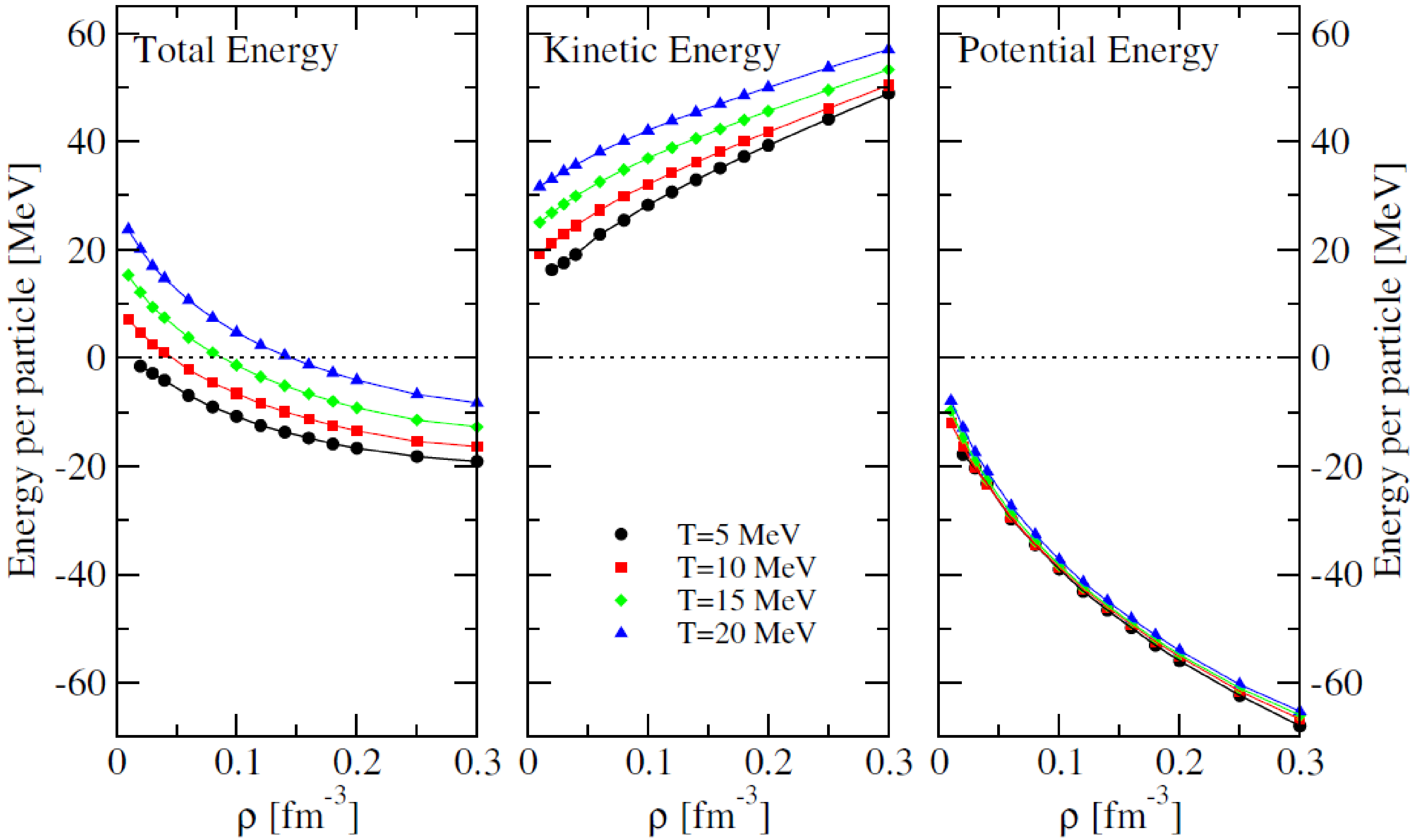
With increasing density, the quasiparticle peak shifts to lower energies with respect to the chemical potential. The situation is the opposite for momentum larger that the Fermi momentum. At the Fermi momentum the peak is approximately located at the chemical potential and its width increases with density. At T=0, and in the absence of pairing correlations the spectral function would have a delta-like quasiparticle peak. The temperature dependence is smaller

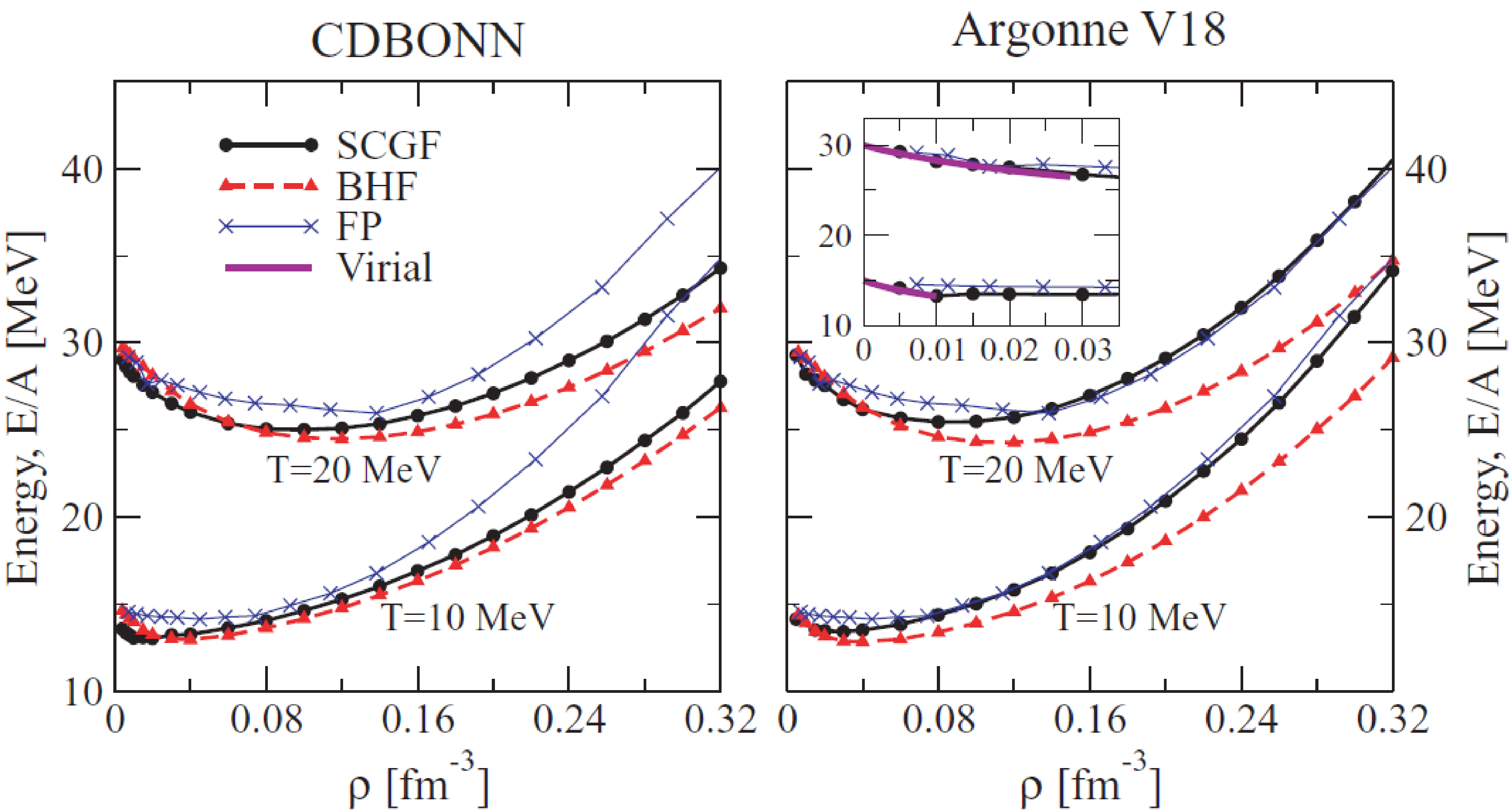


Free energy per particle (solid circles), energy per particle (squares), and the two ways to determine the chemical potential.

As expected, kinetic energy increases with density and for a given density increases with temperature. Tails of n(k) weigthed with k^4

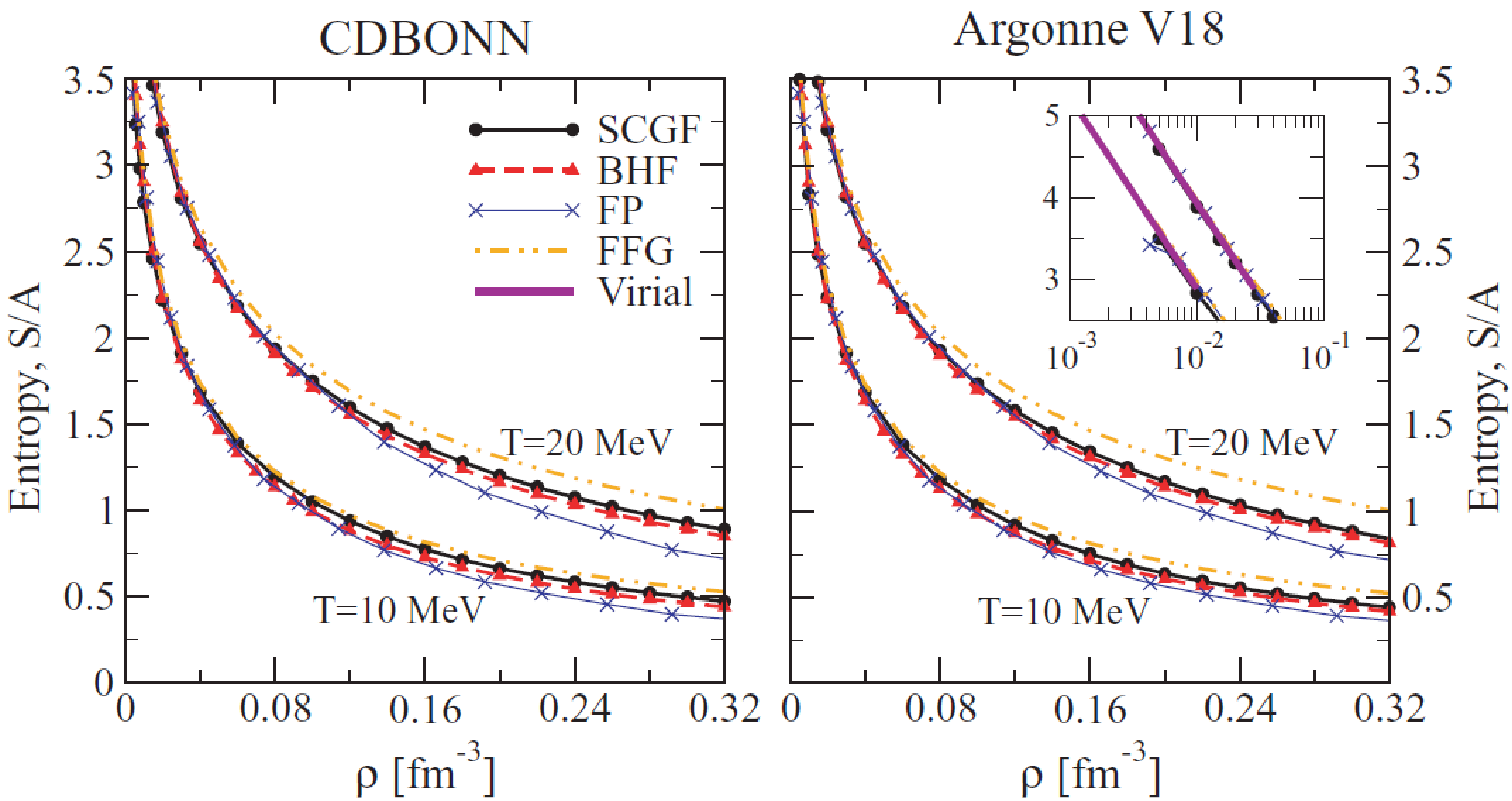
Potential energy gets more attractive when the density increases. Thermal effects are smaller. The balance between the kinetic and potential energies should define a minimum in the total energy.





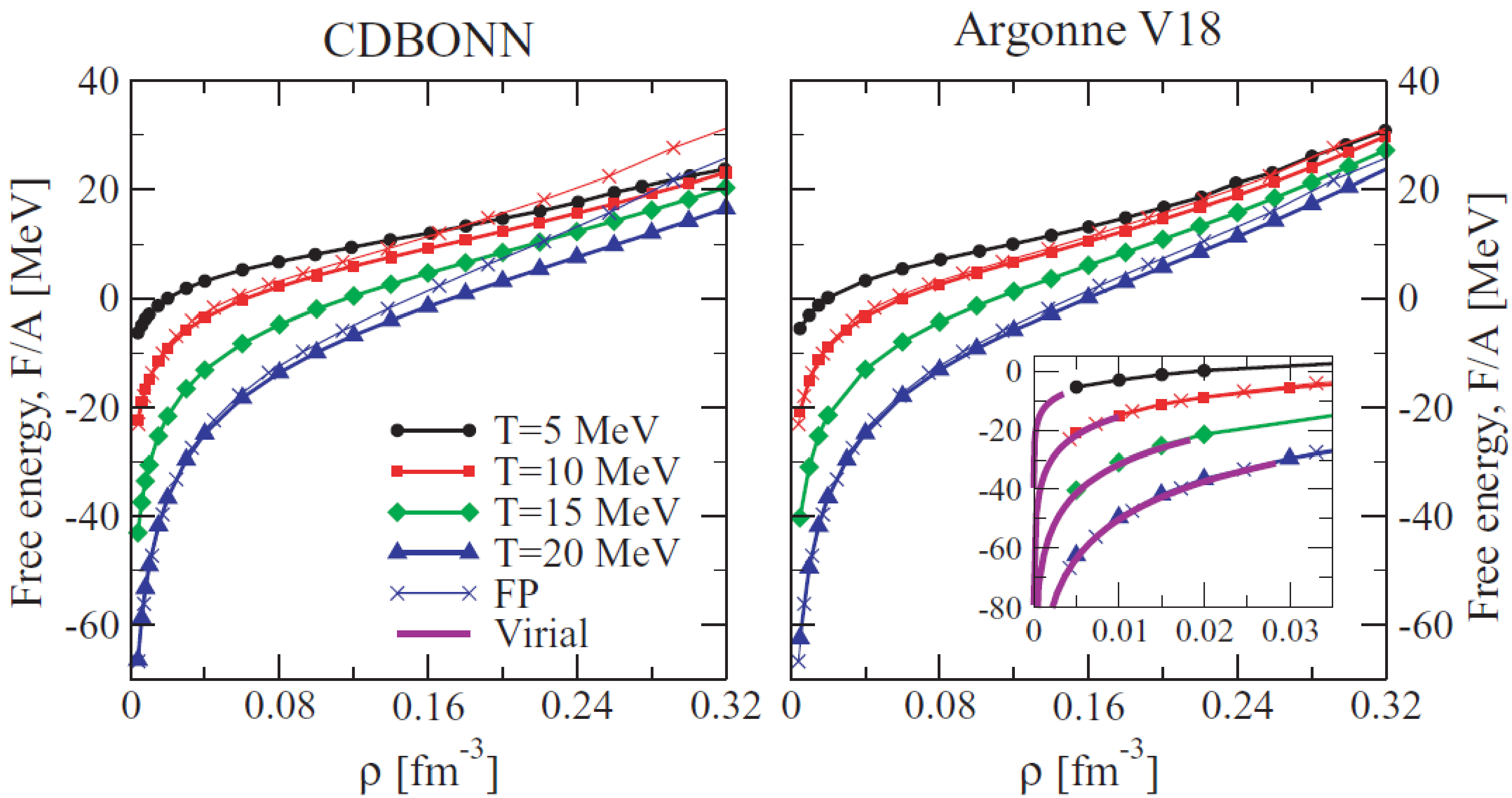
Energy per particle of neutron matter as a function of density for different many-body approaches.

The inclusion of holes in the SCGF leads to more repulsión respect to BHF. The repulsión is larger for higher densities. The effects for Av18 are larger than for CDBONN.

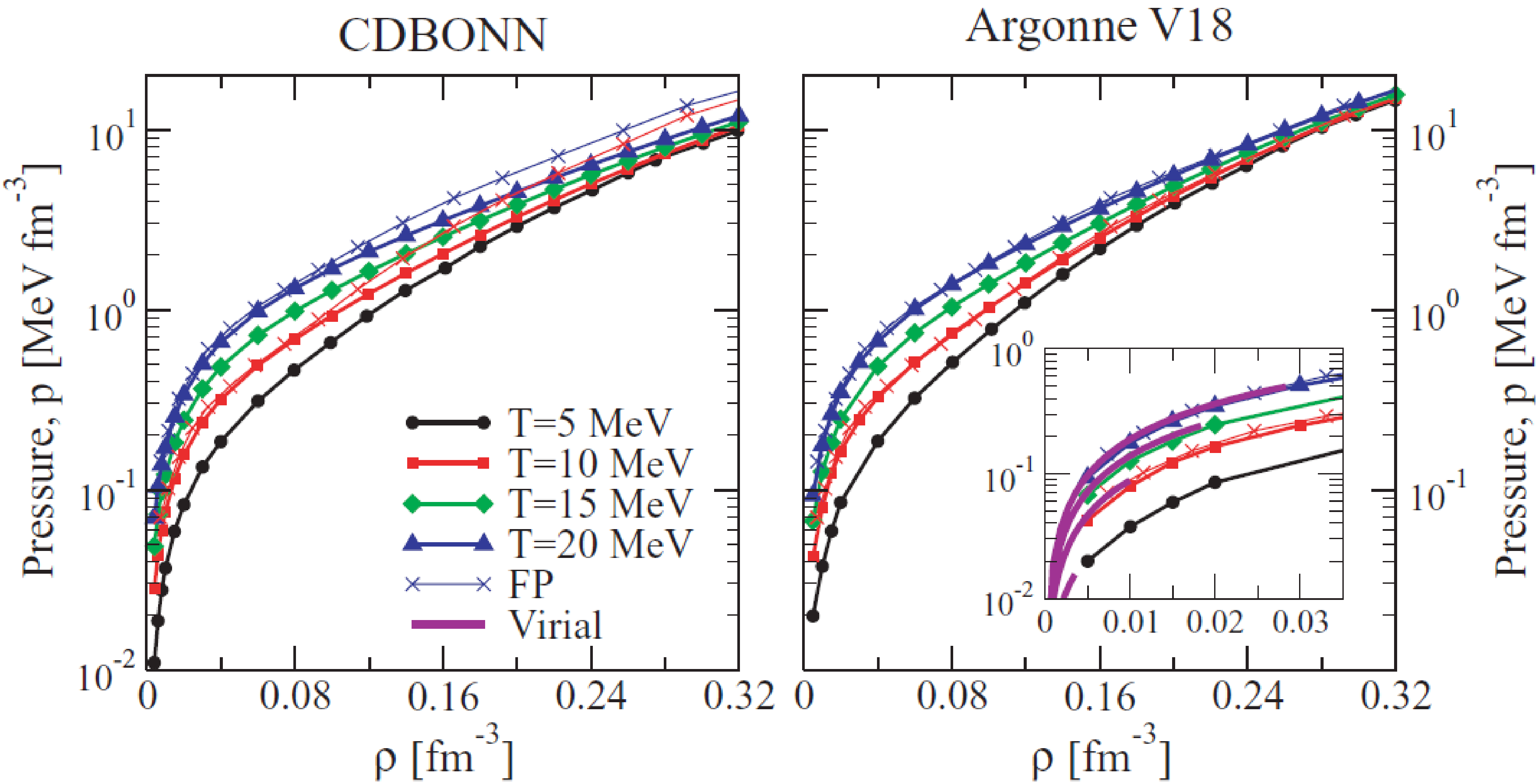


Entropy per particle as a function of density for different approximations

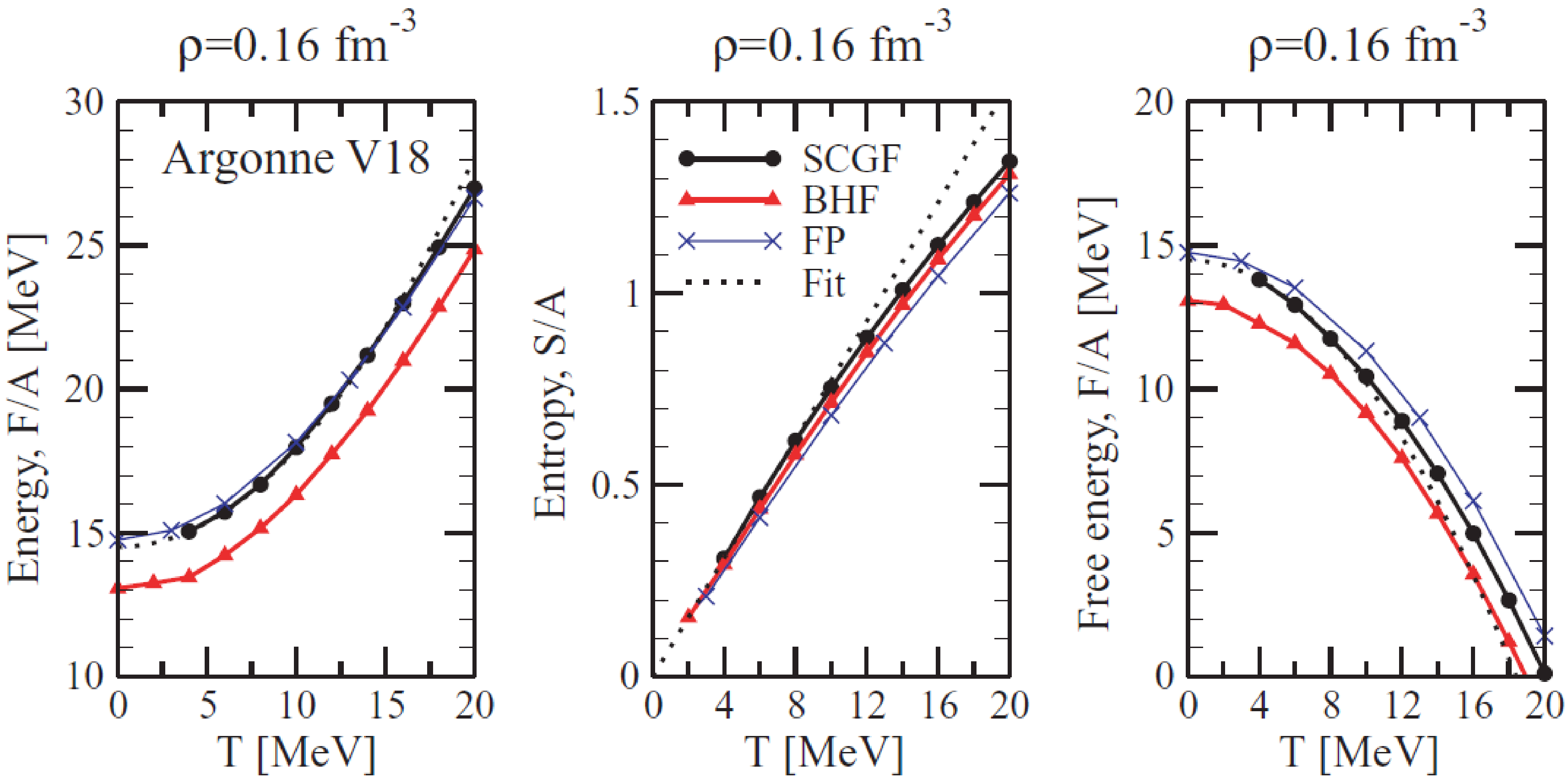
The entropy decreases with density ans increases with temperature.
It does not depend much on the potential or on the many-body method.
At large densities, all methods provide and entropy smaller tan the entropy of the FFG. This is associated with a an effective mass smaller than one.



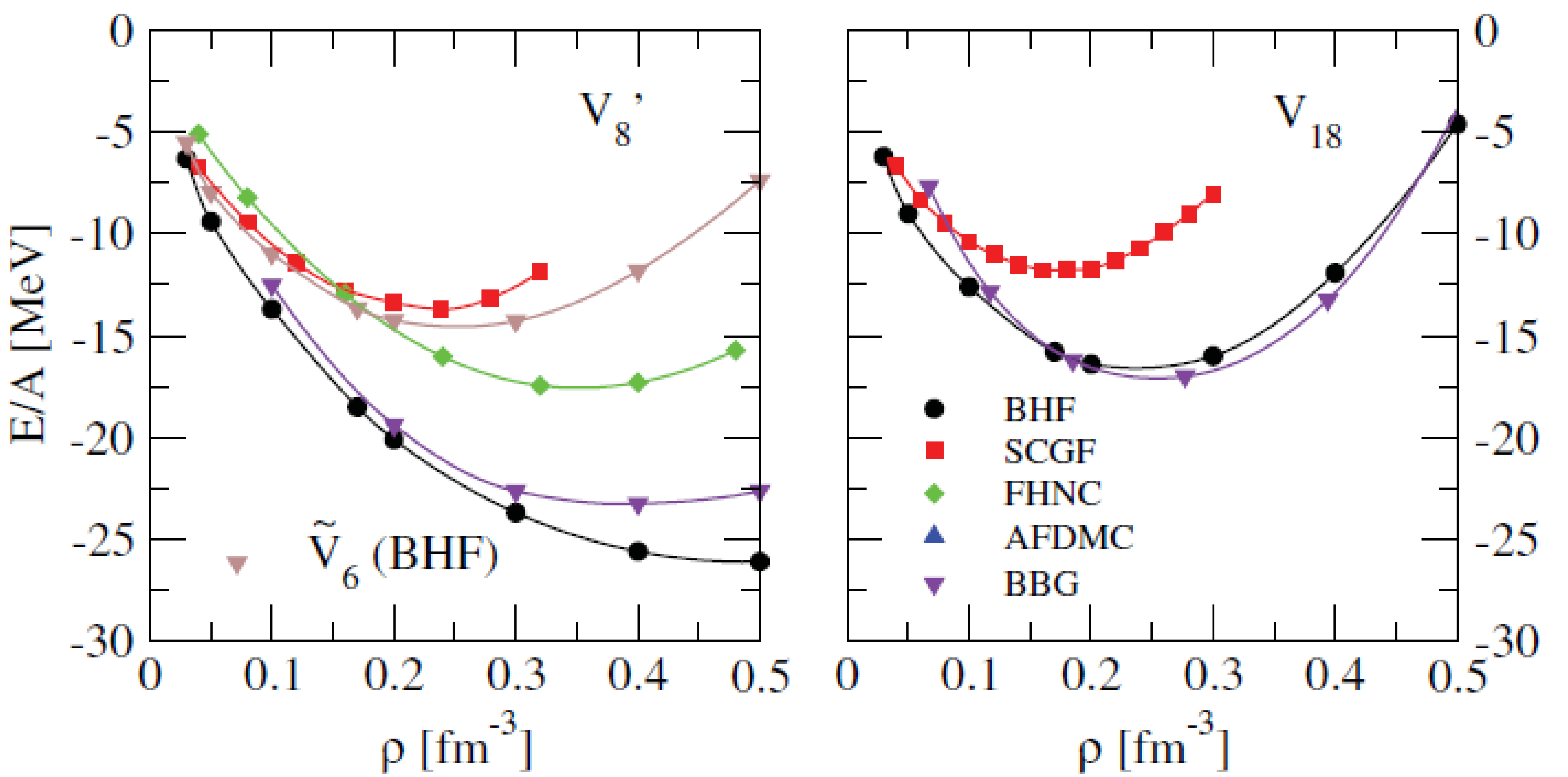
Free energy (f =e – T s) per particle as a function of density for different temperatures for neutron matter .
At low densities (for Argonne) we compare with the virial expansion.
At very low densities the free energy becomes very negative, because the entropy term dominates over the energy contribution.



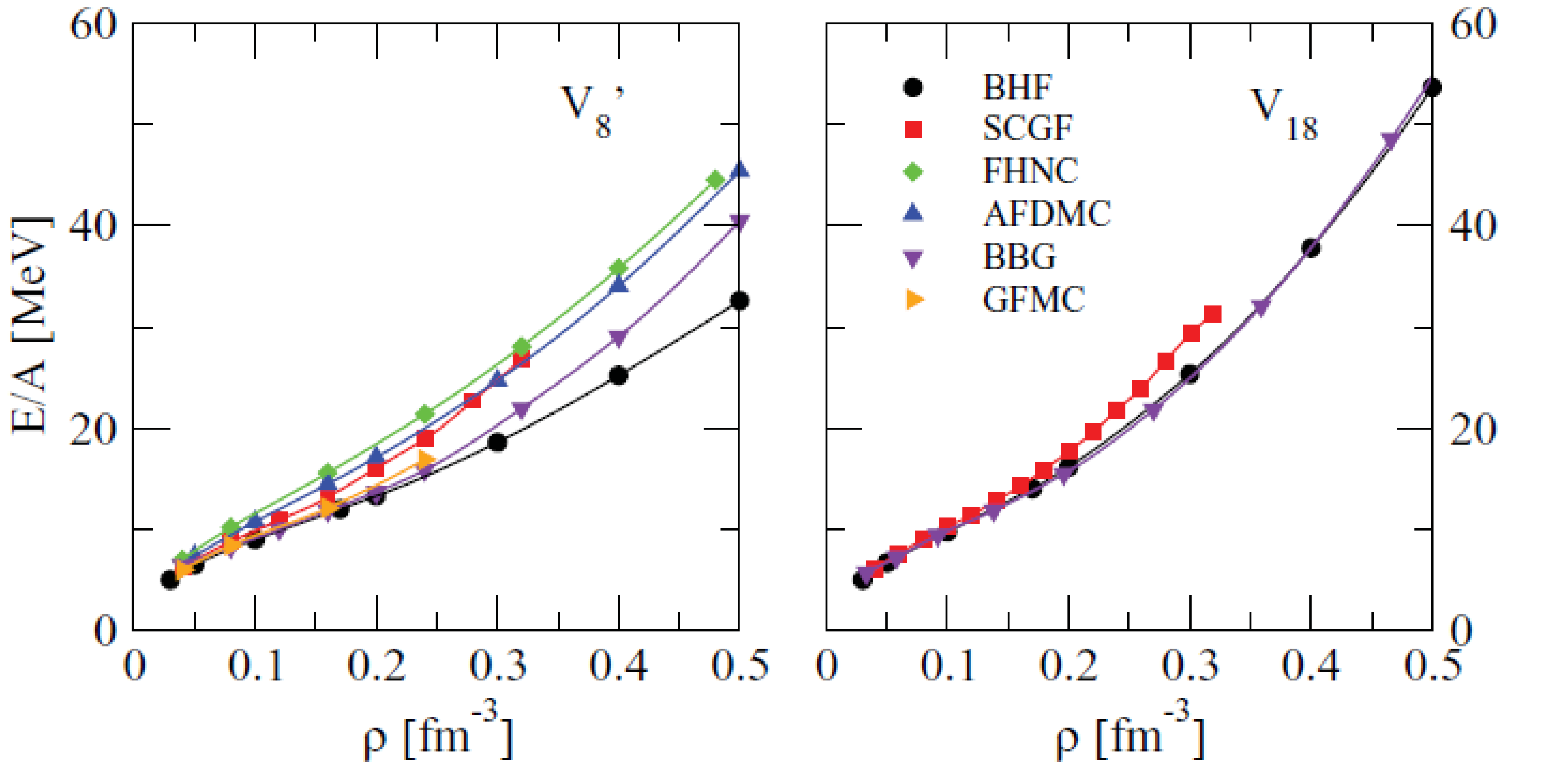
Equation of state (EoS) of neutron matter for different temperatures.
CDBONN produces a softer EoS.
At low density, there is a good agreement with the virial expansion.
As expected, at a fixed temperature, pressure increases with density and at a given density increases with temperature.



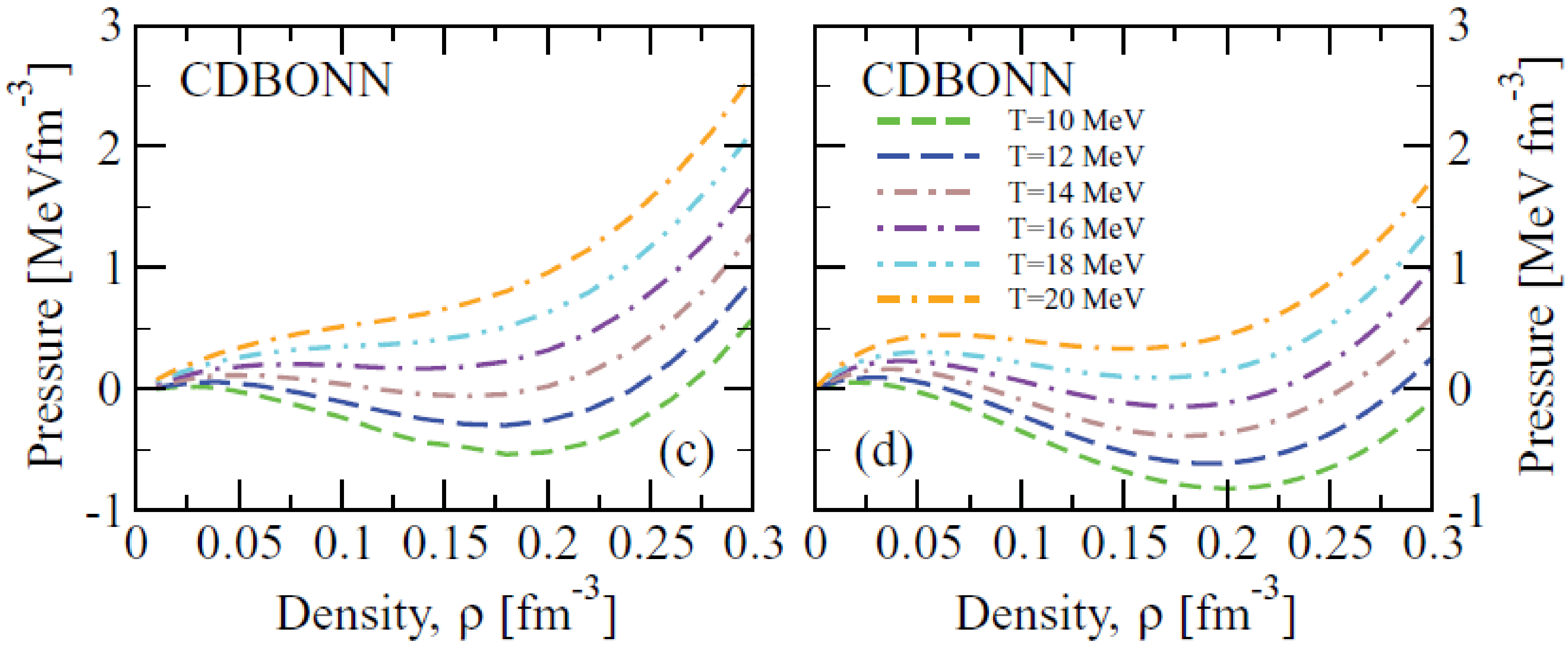
Thermodynamical properties at a given density as a function of temperature. The energy and the entropy increase and the free energy decreases. SCGF is more repulsive than BHF. The entropies are similar for the different approaches.



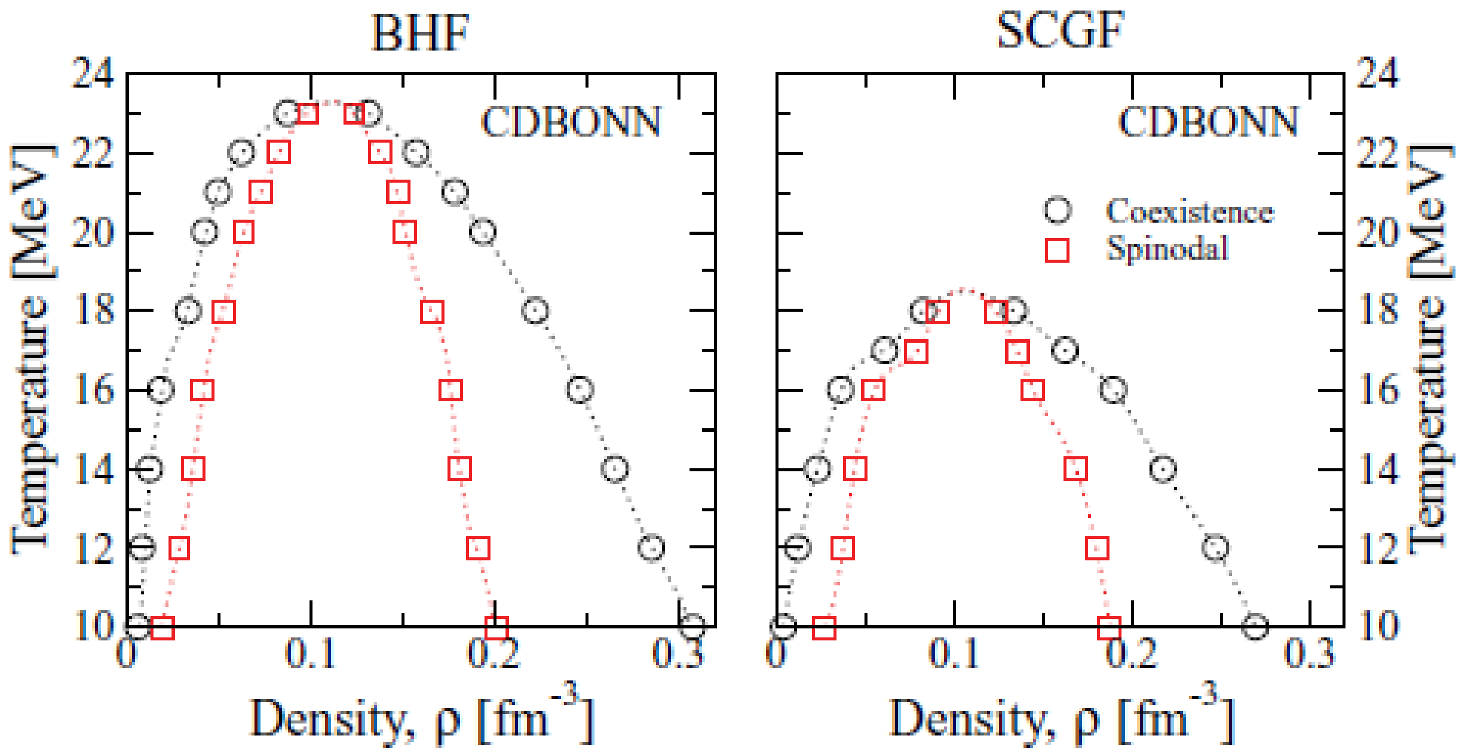
E/A [MeV]
ρ [fm^{-3}]
V_8'
V_{18}
$\tilde{V}_6$ (BHF)
BHF
SCGF
FHNC
AFDMC
BBG
0
-5
-10
-15
-20
-25
-30
0.1
0.2
0.3
0.4
0.5



E/A [MeV]
ρ [fm^{-3}]
V_8'
V_{18}
BHF
SCGF
FHNC
AFDMC
BBG
GFMC



Pressure as a function of density for different temperatures.
Right panel stands for BHF and the left panel corresponds to SCGF.
The shape of the isotherms indicates the existence of a liquid-gas phase transition. In BHF, T_c is larger than 20 MeV while in SCGF is smaller.



Coexistence and spinodal lines for symmetric nuclear matter in the SCGF and BHF approaches.
The plot is for temperatures larger tan 10 MeV. At T=0, the liquid at Saturation density is in equilibrium with a gas at zero density.

In the next lecture we will consider asymmetric systems, with more neutrons than protons.

We will pay attention to the momentum distributions, specially to the high-momentum tails of n(k).

We will also study the symmetry energy and its derivative respect to density at the saturation density.