DOING PHYSICS WITH MATLAB

THE FINITE DIFFERENCE METHOD FOR THE NUMERICAL ANALYSIS OF SERIES RCL CIRCUITS

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Matlab Download Directory

CNsRCL.m

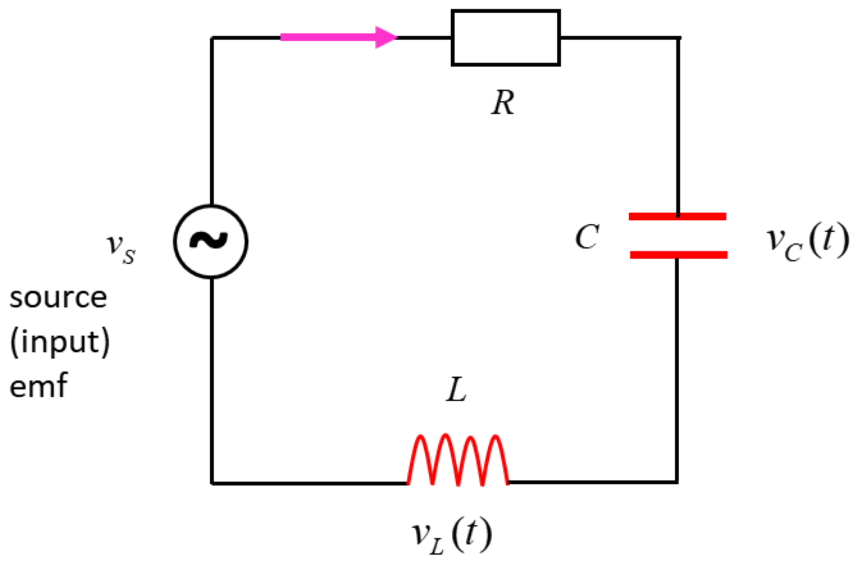
Computation of voltages, current, and energies for series RCL circuits using the finite difference method. The Matlab function findpeaks is used to estimate frequencies (periods) and phases.

An interesting circuit is obtained by connecting a resistor, capacitor and inductor in series with a source (input) emf (figure 1). The behaviour of the circuit is like an object at the end of a spring - **it oscillates**. There is a continual exchange of energy between the energy source and the energies stored in the capacitor and inductor. In a mechanical system, an object oscillates back and forth around an equilibrium position. In the electrical circuit, it is the charge that oscillates. The oscillating charge produces an alternating current and alternating voltage drops across the resistor, capacitor and inductor. The frequency of the oscillation depends only upon the values of the capacitance C and inductance L. This **natural frequency** f_0 (**resonance frequency**) is given by

(1) $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

The response of the series RCL circuit is that of a damped harmonic oscillator. The damping being dependent upon the resistance R. The greater the value of the resistance, the greater the damping effect.

$$i_S(t) = i_R(t) = i_C(t) = i_L(t)$$



$$\frac{dv_C(t)}{dt} = \frac{i_C(i)}{C} \quad \rightarrow \quad v_C(t+\Delta t) = v_C(t) + i_C(t)\ dt\ /\ C$$

$$\frac{di_L(t)}{dt} = \frac{v_L(i)}{L} \quad \rightarrow \quad i_L(t+\Delta t) = i_L(t) + v_L(t)\ dt\ /\ L$$

Fig. 1. Series RCL circuit which can be modelled using the finite difference method.

The response of the series RCL circuit can be computed by using Kirchhoff's Voltage and Current Laws and approximating the voltage across the capacitor and current through the inductor using the finite difference method.

The script **CNsRCL.m** is used to model the series RCL circuit. In the script, the first letter of a variable that is a function of time or a complex variable is a lowercase letter. A variable that start with an uppercase letter is independent of time or a real quantity. For example, **vS** is the variable for the input emf and is calculated at each time step, whereas the variable **VS** is the peak value of the source emf.

A **step function** (OFF/ON) or a **complex sinusoidal function** can be selected as the source emf using the variable **flagV** (flagV = 1 for a step function or flagV = 2 for a sinusoidal function). The source emf can be changed within the switch/case script. The time scales for a simulation are also set within the switch/case script.

The complex sinusoidal function is used for the source emf is

```
vS = VS .* exp(1j*(w*t - pi/2));
```

It is better to use complex functions for some of the variable, because the complex function contains information of both the magnitude and phase of the variable. The real part of a complex function gives its actual value. The actual emf that is used is a sine function because we assume the capacitor is initially uncharged

(2) $$v_S(t) = V_S \sin(\omega t)$$

To start the computational procedure, the initial conditions must be specified at time step #1.

```
%   Time Step #1
    vR(1)   = vS(1);
    vC(1)   = 0;
    vL(1) = vS(1) - vR(1) - vC(1);
    iS(1) = vR(1) / R;
    iS(1) = iS(1) + vL(1) * dt / L;
    vC(1) = vC(1) + iS(1) * dt / C;
    vR(1) = iS(1) * R;
    vL(1) = vS(1) - vR(1) - vC(1);
```

The values of the voltage and current parameters are calculated by implementing the finite difference method. Note: the voltage across the capacitor uses an average value of the current over two time steps to improve the accuracy of the numerical approximation (half-step method).

```
% Time Steps #2 to #N
 for c = 2 : N
  iS(c) = iS(c-1) + vL(c-1) * dt/L;
  vR(c) = iS(c) * R;
  vC(c) = vC(c-1) + 0.5*(iS(c)+iS(c-1)) * dt / C;
  vL(c) = vS(c) - vR(c) - vC(c);
 end
```

For accurate results, the time interval dt should be chosen so that it is much smaller the period of the natural oscillation

(3) $$dt \ll 2\pi\sqrt{LC}$$

```
% Resonance Frequency f0, period T, time step dt
   f0 = 1/(2*pi*sqrt(L*C));
   T0 = 1/f0;
   dt = T0 /1000;
```

From the values of R, C and L, the impedances Z of the circuit elements for the sinusoidal source emf are calculated as shown in the Table.

```
% Impedance calculations
       XC = 1/(w*C); ZC = -1j*XC;
       XL = w*L;     ZL = 1j*XL;
                     ZR = R;
       Z = ZR + ZC + ZL;
```

Energy is dissipated by a current through the resistor and energy is stored in the electric field of the capacitor plates, and stored in the magnetic field surrounding the coil of the inductor. The power $p(t)$ and energy $u(t)$ as functions of time t can easily be computed.

```
% Powers and energy
   pS = real(vS) .* real(iS);
   pR = real(vR) .* real(iS);
   pC = real(vC) .* real(iS);
   pL = real(vL) .* real(iS);
```

```
uS = zeros(1,N); uR = zeros(1,N);
uC = zeros(1,N); uL = zeros(1,N);
for c = 2 : N
    uS(c) = uS(c-1) + pS(c)*dt;
    uR(c) = uR(c-1) + pR(c)*dt;
    uC(c) = uC(c-1) + pC(c)*dt;
    uL(c) = uL(c-1) + pL(c)*dt;
end
```

Our series RCL circuit is complicated. We can not simply add the voltages across the resistor, capacitor and inductor because of the phase differences between these voltages. However, in the modelling, we use a complex exponential function to simulate a real sine function source voltage. The computed values for the circuit current and voltages are all computed as complex functions. So, these complex functions contain information of the magnitudes and phases. The script below shows the calculation of the phases at a time step given by the variable **nP**. By changing the value of **nP**, you can see the phases at different times.

```
% Phases voltage: phi and current theta  at time
step nP  [degrees]
    nP = N-500;

    phiS   = rad2deg(angle(vS(nP)));
    phiR   = rad2deg(angle(vR(nP)));
    phiC   = rad2deg(angle(vC(nP)));
    phiL   = rad2deg(angle(vL(nP)));
```

```
    thetaS = rad2deg(angle(iS(nP)));
    phiSR = phiS - phiR;
```

The response of the circuit may result in oscillations. The period T_{Peaks} of the oscillation can be approximated by finding the time intervals between peaks using the Matlab function **findpeaks** as shown in the code in the Table.

```
% Find peaks in vS and corresponding times
%   Calculate period of oscillations
%   May need to change number of peaks for
    estimate of period: nPeaks
    [iS_Peaks, t_Peaks] = findpeaks(real(iS));
    nPeaks = 3;
    T_Peaks = (t(t_Peaks(end))- t(t_Peaks(end-
nPeaks)))/(nPeaks);
    f_Peaks = 1 / T_Peaks;
```

If no peaks are found, you may get an error message. If this happens, simply set the statements to comments using % and set f_Peaks = 0 and T_peaks = 0.

A summary of the input and calculated parameters is displayed in the Command Window. The results of the modelling are displayed graphically in a series of plots as shown in the following simulations.

RESPONSE TO A STEP FUNCTION OFF / ON

The response of the RCL circuit is like that of a mass at the end of a string.

Natural frequency for the series RCL is

(1) $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Natural frequency of oscillation of mass / spring system is

$$f_0 = \frac{1}{2\pi\sqrt{m/k}}$$

In this analogy

$$L \leftrightarrow m \quad C \leftrightarrow 1/k \quad R \leftrightarrow b$$

where b is the damping coefficient of the velocity term in the equation of motion of the mass on a spring.

When an object attached to a spring is disturbed, is motion can often be classified as underdamped, critically damped or overdamped. **Critical damping** provides the quickest approach to zero amplitude for a damped oscillator. With less damping (**underdamping**) it reaches the zero position more quickly, but oscillates around it. With more damping (**overdamping**), the approach to zero is slower. For our series RCL circuit, the damping is determined by the value of the resistance.

The **critical damping resistance** is given by

$$R_{critical} = 2\sqrt{\frac{L}{C}}$$

When $R < R_{critical}$ the system is underdamped and when $R > R_{critical}$ the system is overdamped.

When there is underdamping $\left(R < R_{critical}\right)$, the system vibrates at its natural frequency until the oscillations die away. Any sudden changes in the source emf may produce a ringing effect, where the current oscillates at the resonant (natural) frequency determined by the values of C and L as given by equation 1. The amplitude of the current dies away exponentially.

Simulation 1:

Step Function OFF / ON source emf

Underdamping $R = 10\ \Omega \quad R_{critical} = 237\ \Omega$

Circuit parameters and numerical results are displayed in the

Command Window

Resistance R = 1.00e+01 ohms

Capacitance C = 1.000e-06 F

Inductance L = 1.400e-02 H

Source emf: step Function OFF/ON

Peak emf VS = 1.00e+01 V

Resonance Frequency f0 = 1.345e+03 Hz

Resonance Period T0 = 7.434e-04 s

time increment dt = 7.434e-07 s

dt / T = 1.00e-03

Frequency of peaks f_Peaks = 1.344e+03 Hz

Period of peaks T_Peaks = 7.439e-04 s

Figure 2 shows the plots of the current as a function of time for two underdamped systems. When the resistance R is increased from R = 10 Ω to R = 40 Ω, the oscillations die away more quickly due to the increase in damping. Figure 3 shows the voltages across the resistor, capacitor and inductor.

The natural frequency f_0 of the oscillation calculated from equation 1 is

$$f_0 = 1345 \text{ Hz}$$

The value of the natural frequency f_{peaks} from the model using the **findpeaks** function is

$$f_s = 1344 \text{ Hz}$$

So, we have excellent agreement between the two values.

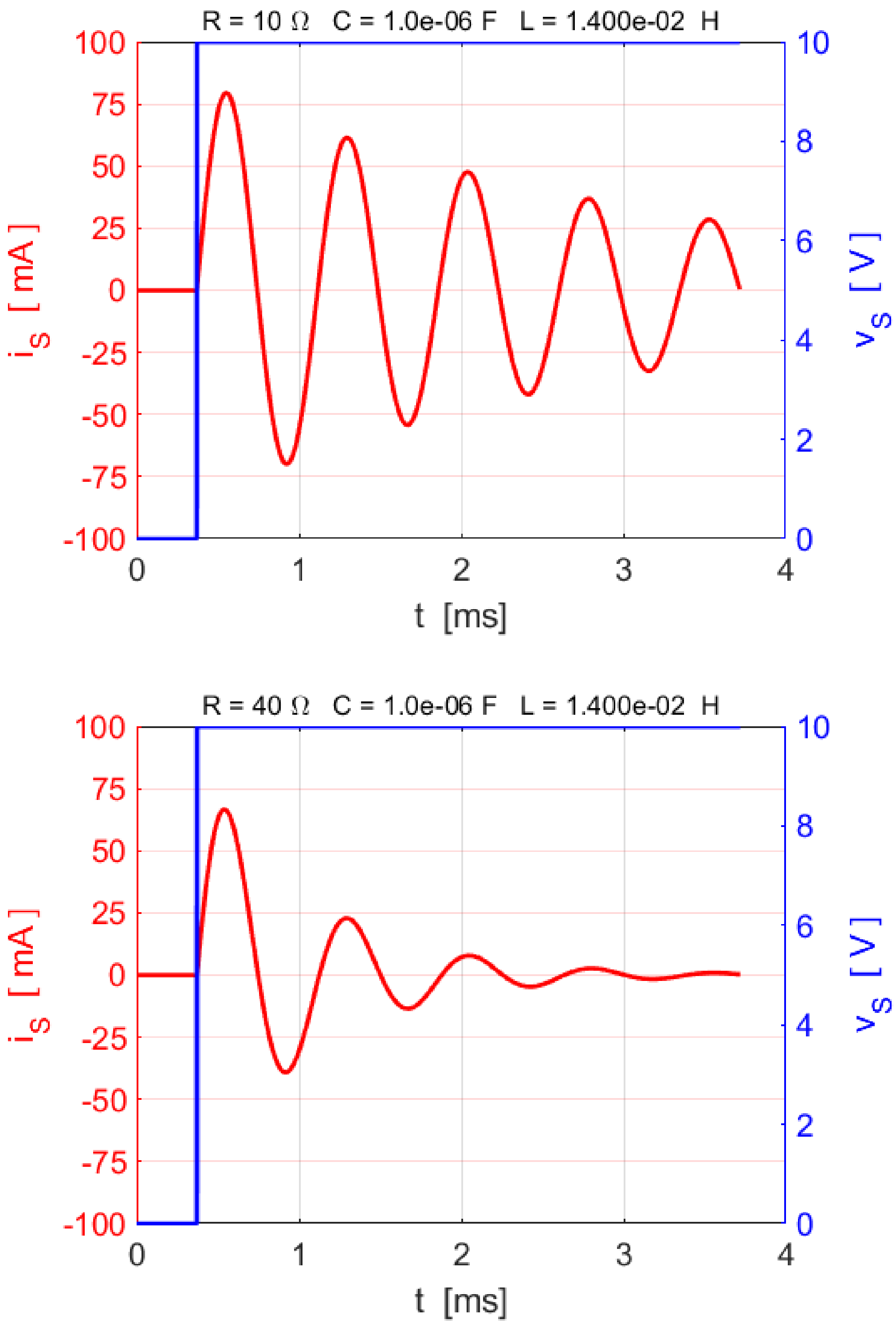


Fig.2.   The ringing effect of the step function source emf. The oscillations die way exponentially. The larger the resistance R, the more rapidly the oscillations decrease.

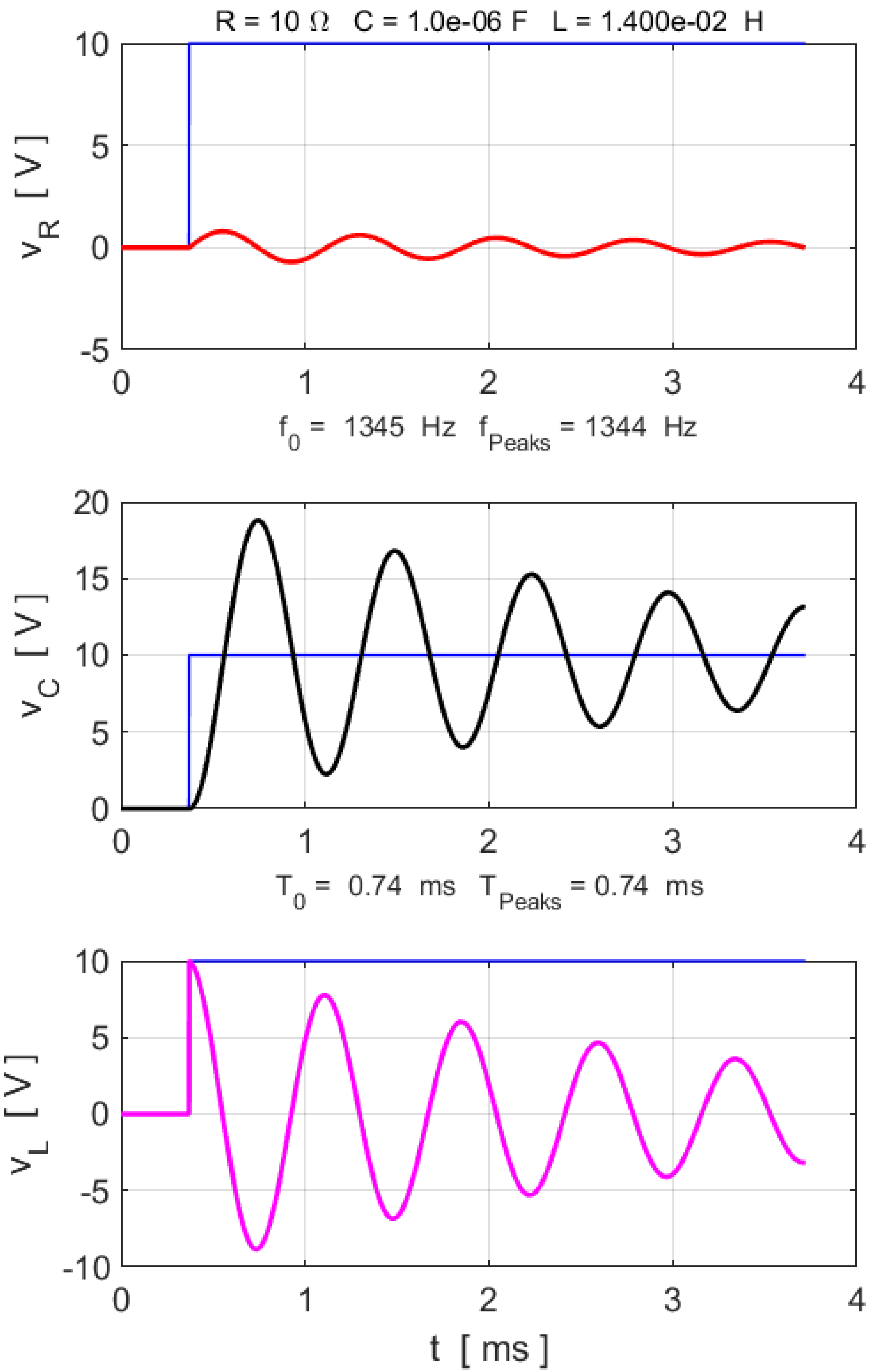


Fig. 3. The voltages as function of time. The ringing effect is very noticeable from the plots. The blue curves represents the source emf.

We can see clearly from figure 3 that there is a π rad phase difference between for the voltage across the capacitor and the voltage across the inductor (a peak in v_C corresponds to a trough in v_L). Through careful examination of figure 3, we can state that

$$v_R = V_R \sin(\phi) \quad v_C = V_C \sin(\phi - \pi/2) \quad v_L = V_L \sin(\phi + \pi/2)$$

Hence, we can conclude, The voltage across the capacitor lags the voltage across the resistor by $\pi/2$ rad; the voltage across the inductor leads the voltage across the resistor by $\pi/2$ rad; and the voltage across the inductor leads the voltage across the capacitor by π rad.

We can also get the phase values from the Command Window using the findpeaks command. The findpeaks command is used to find the indies for the times of the least peak in the voltages.

[a b] = findpeaks(real(vR)) →

a = 0.7942 0.6149 0.4761 0.3686 0.2854

b = 782 1782 2783 3784 **4784**

The times for the last peaks are

resistor t(4784) = 3.5559 ms

capacitor t(5041) = 3.7469 ms

inductor t(4527) = 3.3648 ms

We can calculate the period from the values of b given from the findpeaks function

period T = (t(4784) -t(1782)) /3 (time between 3 peaks)

T = 0.7439 ms

We can now find the phases by comparing the time difference between peaks and the period and expressing the phase angle in degrees.

Phase difference between v_C and v_R

$$\phi_C = (360)\left(\frac{3.7469 - 3.5559}{0.7439}\right)^{\mathrm{o}} = 92^{\mathrm{o}}$$

Phase difference between v_L and v_R

$$\phi_L = (360)\left(\frac{3.3648 - 3.5559}{0.7439}\right)^{\mathrm{o}} = -92^{\mathrm{o}}$$

Phase difference between v_L and v_C

$$\phi_{LC} = 92^{\mathrm{o}} - \left(-92^{\mathrm{o}}\right) = -184^{\mathrm{o}}$$

Figures 4 and 5 shows the exchanges of energy occurring in the circuit as functions of time. The emf source provides energy to the circuit. When a current passes through the resistance, energy is absorbed and dissipated as thermal energy resulting in an increase in temperature of the resistor. The capacitor stores energy as it charges and supplies energy to the circuit as it discharges. For our step function emf input voltage, after the oscillations die away, the capacitor becomes fully charged and stores energy since the capacitor acts like an open circuit, the current falls to zero. The inductor stores in the magnetic surrounding the coil energy as the current through it increases. When the current finally drops to zero, the inductor no longer stores or supplies energy to the circuit. For the power curves as functions of time, when the power is positive, energy is either dissipated or stored. When the power is negative, the stored energy is returned to the circuit.

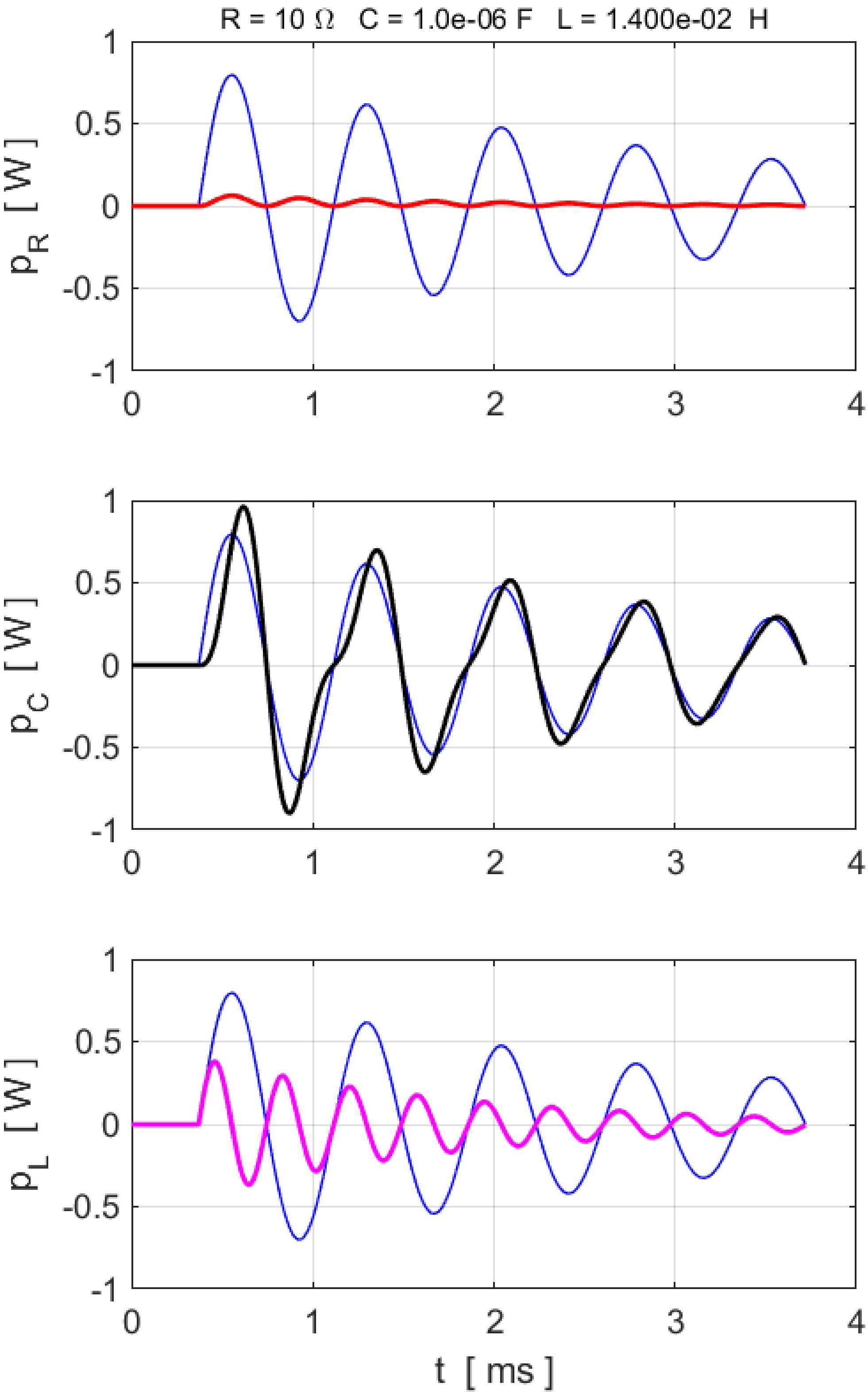


Fig. 4. The power absorbed or supplied by the circuit elements. The blue curves represent the source emf.

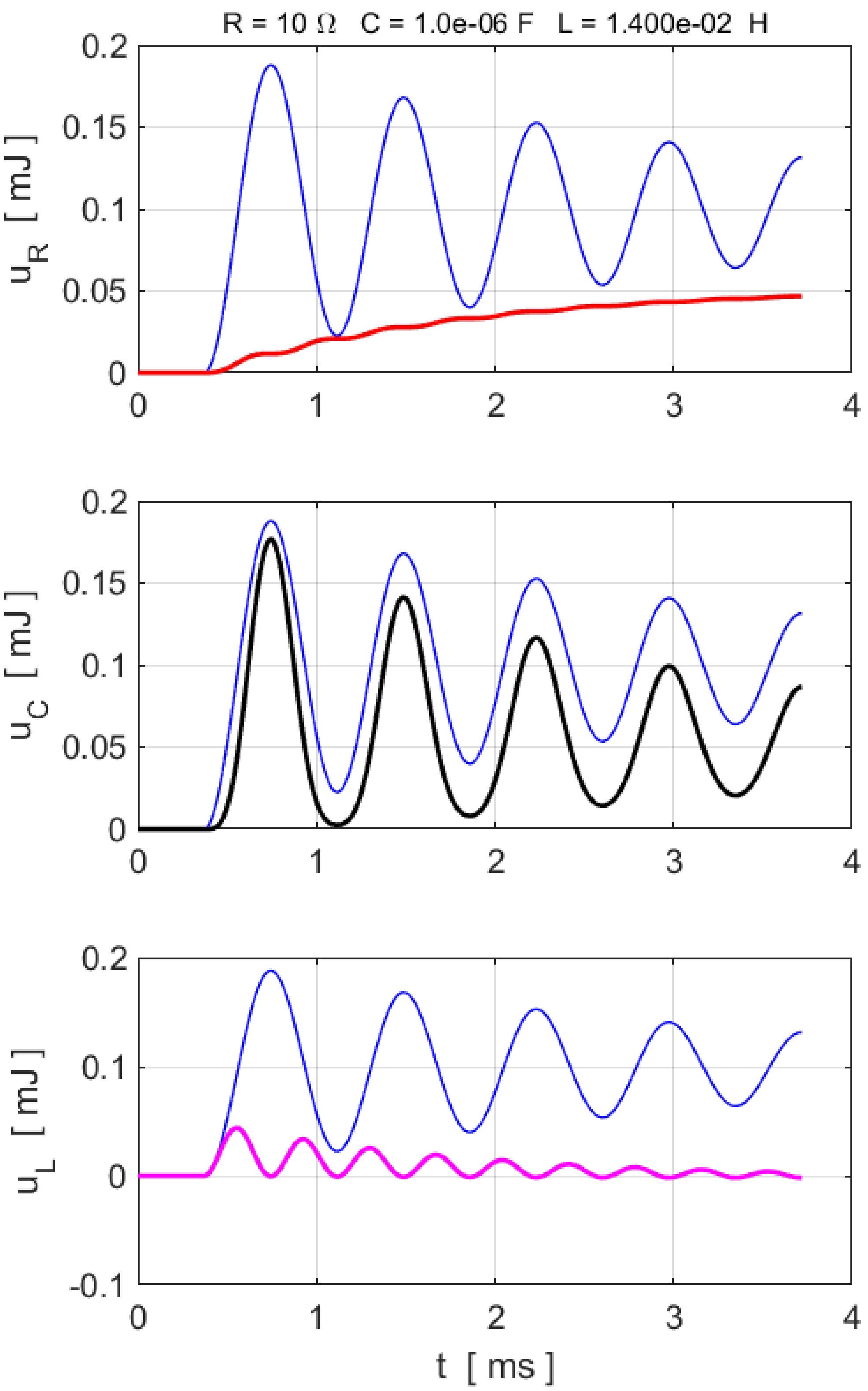


Fig. 5. The energy exchanges in the circuit.

The power is the rate of energy transfer

(4) $$p(t) = \frac{du(t)}{dt}$$

If you "mentally" differentiate an energy plot w.r.t. time you get the power plot as a function of time. For example, in figure 5, the first peak in the energy plot for the capacitor occurs at the time t = 0.74149 ms. At this time, the power p_C is zero.

It is very easy to change any of the parameters in the script, and see immediately, how the response of the circuit changes. For example, figure 6 shows the response when the capacitor value is decreased by a factor of 9. The natural frequency is now 4035 Hz.

$$C \to C/9 \quad f_0 \propto 1/\sqrt{C} \quad \Rightarrow \quad f_0 \to 3 f_0 \quad (3)(1345) = 4035$$

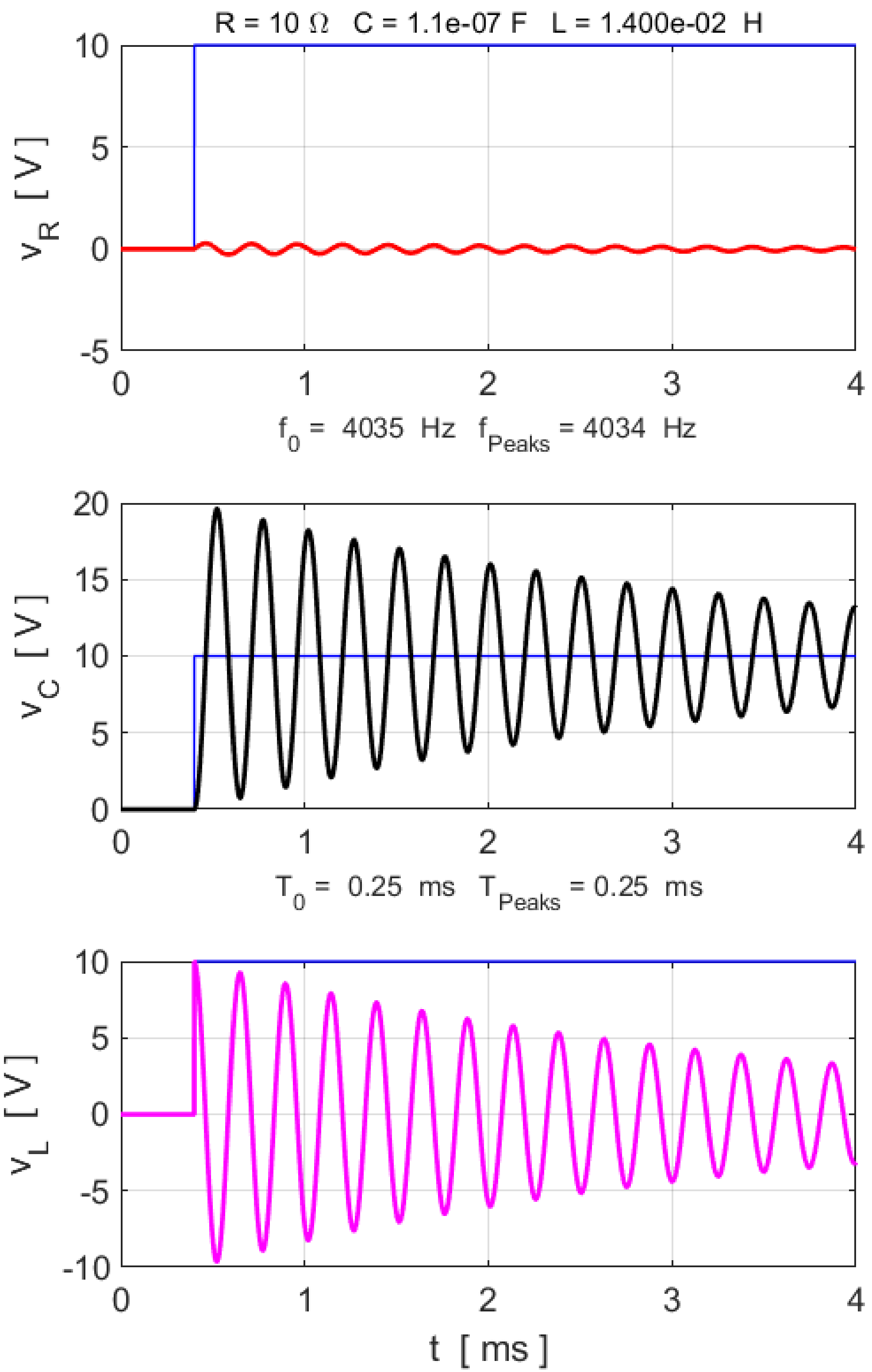


Fig. 6. A decrease in capacitance C results in higher frequency oscillations.

Simulation 2:

Step Function OFF / ON source emf

Underdamping / Critical damping / overdamping

$R_{critical} = 237\ \Omega$

The following figures shows the changes in the damping of the voltages as the resistance of the circuit is increased.

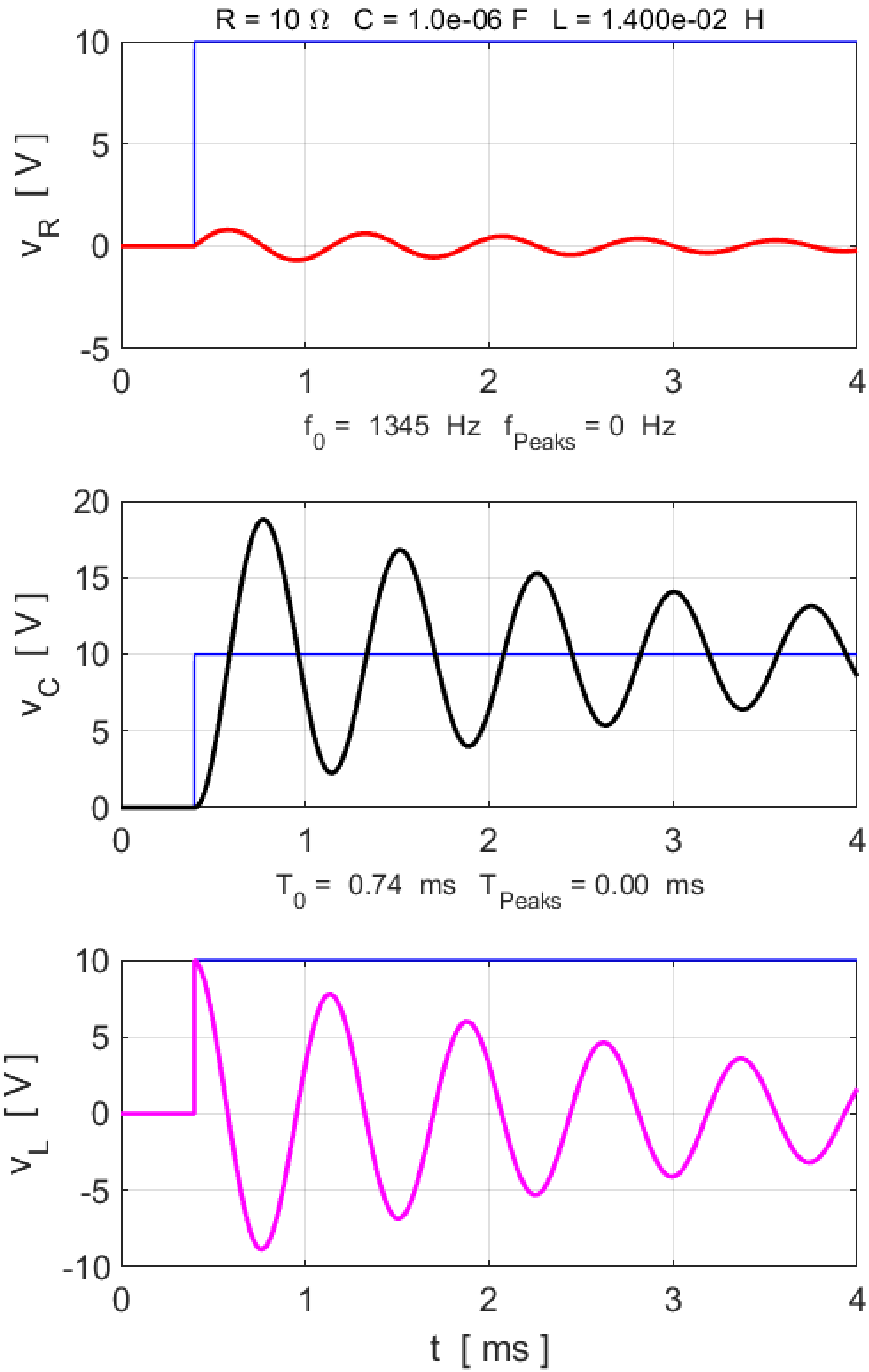


Fig. 7.   Underdamped oscillations.

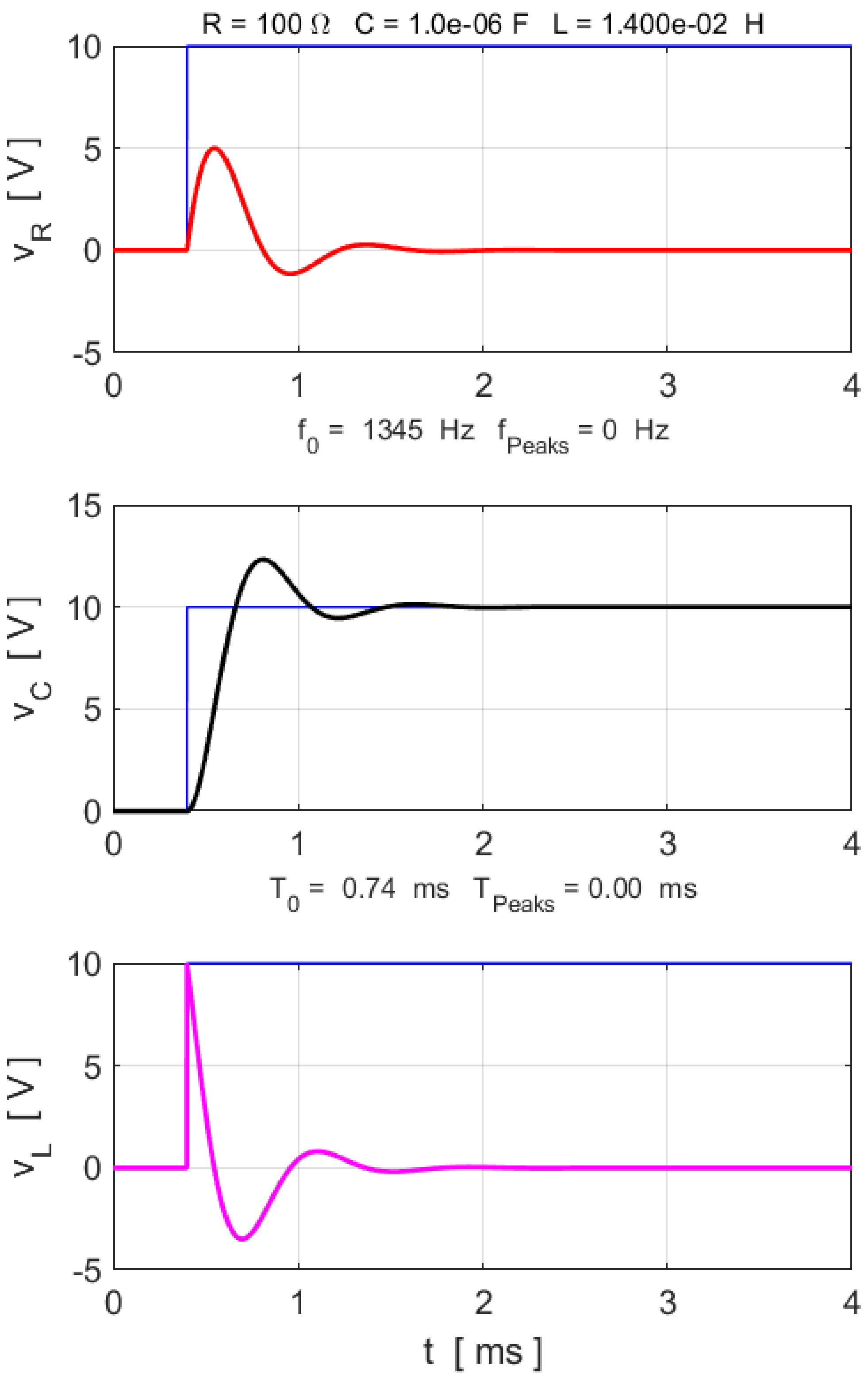


Fig. 8. Heavily underdamped damped oscillations.

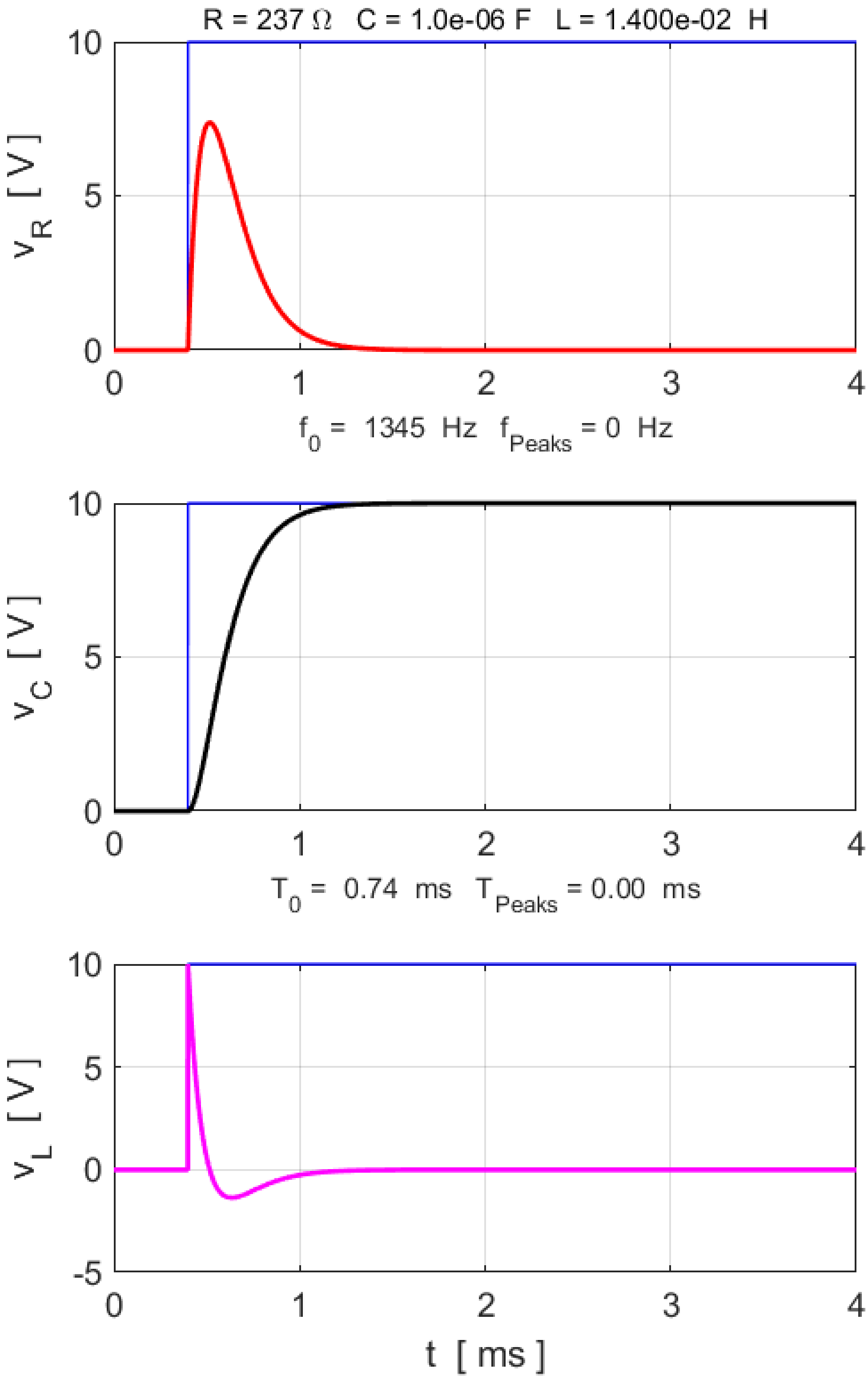


Fig. 10. Critically damped signals – no oscillations. In the script need to comment the lines for findpeaks.

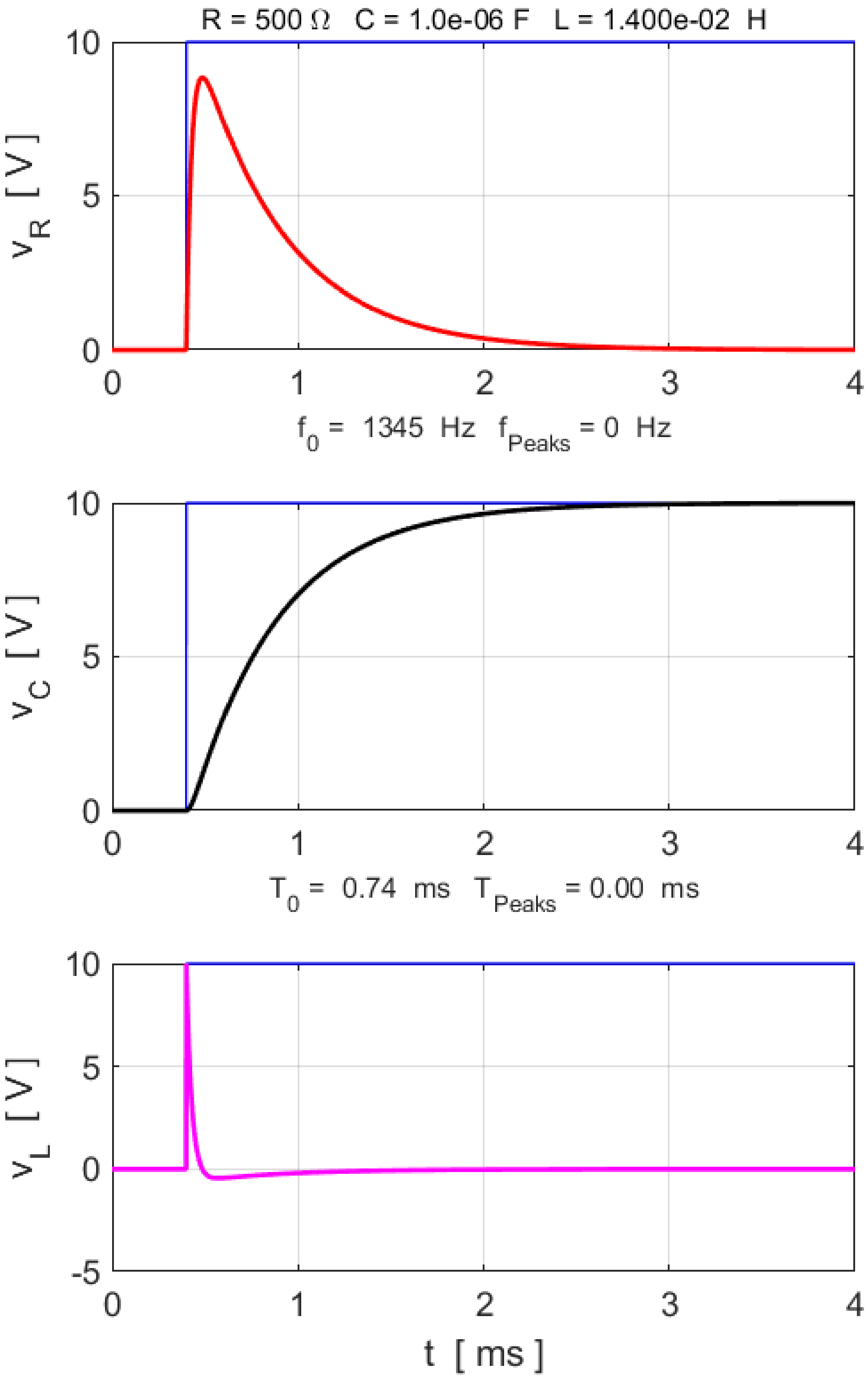


Fig. 10. Overdamped signals – no oscillations. In the script need to comment the lines for findpeaks.

RESPONSE TO A SINUSOIDAL SOURCE VOLTAGE

A mechanical oscillator will vibrate at the driving frequency. As the driving frequency approaches the natural frequency of vibration of the system, the amplitude of the oscillation can be become very large. This phenomenon is called **resonance**. Resonance occurs in a series RCL circuit. The current in the circuit oscillates at the same frequency as the sinusoidal source emf. When the source emf frequency matches the natural frequency as given by equation 1, the amplitude of the oscillation is a maximum for a given amplitude of the source emf.

Simulation 3: Sinusoidal source emf

Figure 11 shows two plots, one where the source frequency is equal to the natural frequency $\left(f_S = f_0 = 1000 \text{ Hz} \right)$ and the second plot, the source frequency higher than the natural frequency $\left(f_S = 1500 \text{ Hz} \right)$.

Warning: It always takes a few cycles before the current (or voltages) to vary sinusoidally with a constant amplitude (peak value). The amplitudes of the sinusoidal current oscillations are:

$$f_S = f_0 = 1000 \text{ Hz} \quad I_S = 249 \text{ mA} \qquad f_S = 1500 \text{ Hz} \quad I_S = 74 \text{ mA}$$

At resonance the total circuit impedance is equal to the resistance. The effects of the capacitor and inductor cancel each other. So, the current in the circuit is

$$I_S = \frac{V_S}{R} = \frac{10}{40}\ \text{A} = 0.25\ \text{A} = 250\ \text{mA}$$

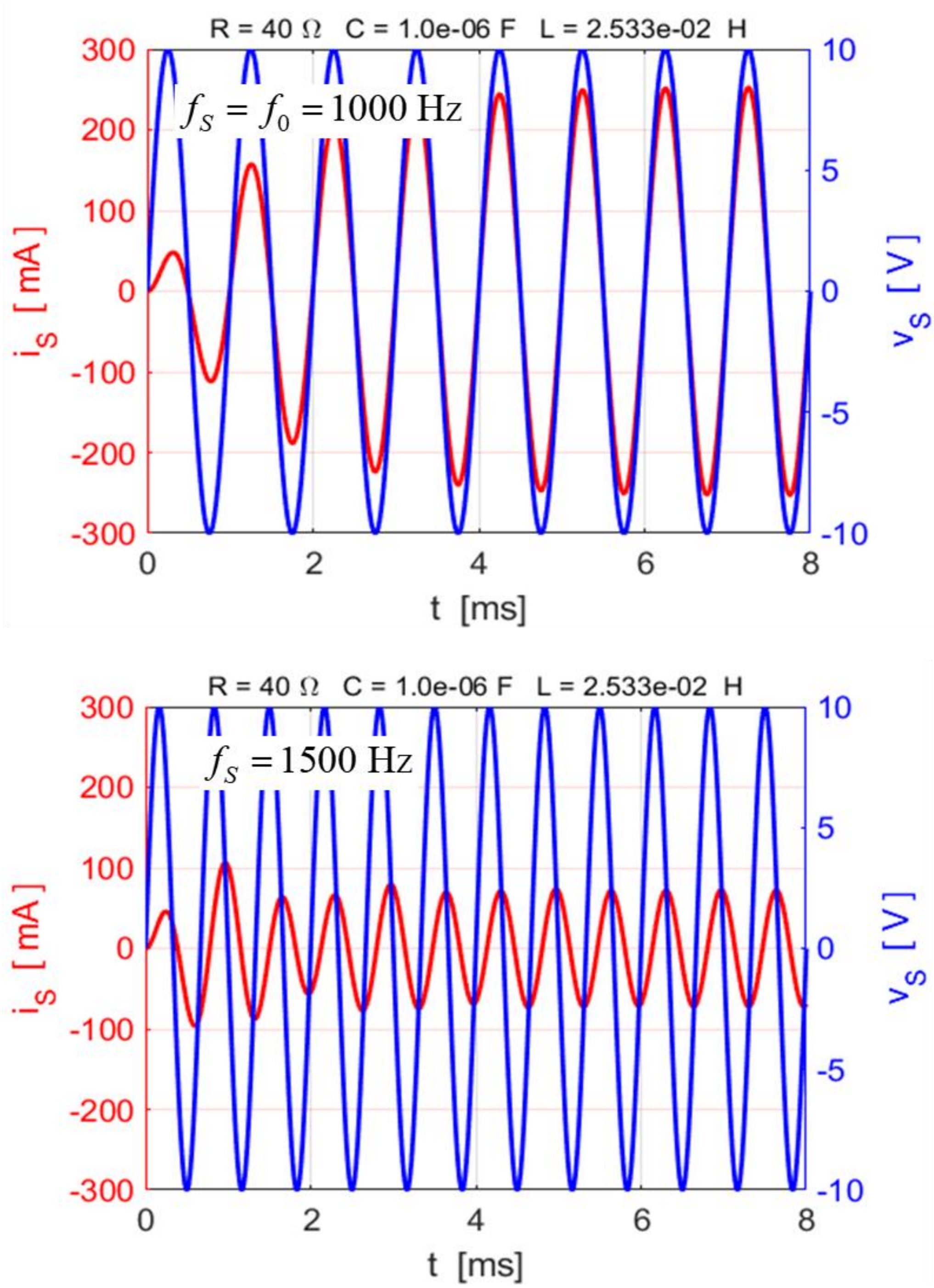


Fig. 11. The current in the circuit is a maximum when driven at the resonance frequency.

Figures 12 shows the voltage plots when the frequency of the complex exponential function is equal to the natural frequency $\left(f_S = f_0 = 1000 \text{ Hz} \right)$. After a few cycles, the effects of the capacitor and inductor cancel. The magnitudes of the voltage across the capacitor is equal to the magnitudes of the voltages across the inductor but they are π rad (180°) out of phase $\left(v_C(t) + v_L(t) = 0 \right)$. So, the voltage across the resistor is identical to the source emf $\left(v_S(t) = v_R(t) \right)$.

It is better to use a complex exponential function rather than the sine function for the source emf as we can extract both the magnitude and phase from it. This means that we can compare the phases across the resistor, capacitor, inductor and current with the phase of the source emf. Figure 9 shows the phasor diagram when $f_S = f_0 = 1000 \text{ Hz}$ for the voltages at one instant when t = 9.50 ms and at a slightly later time t = 9.70 ms. The voltage of the source emf is in phase with voltage across the resistor at resonance.

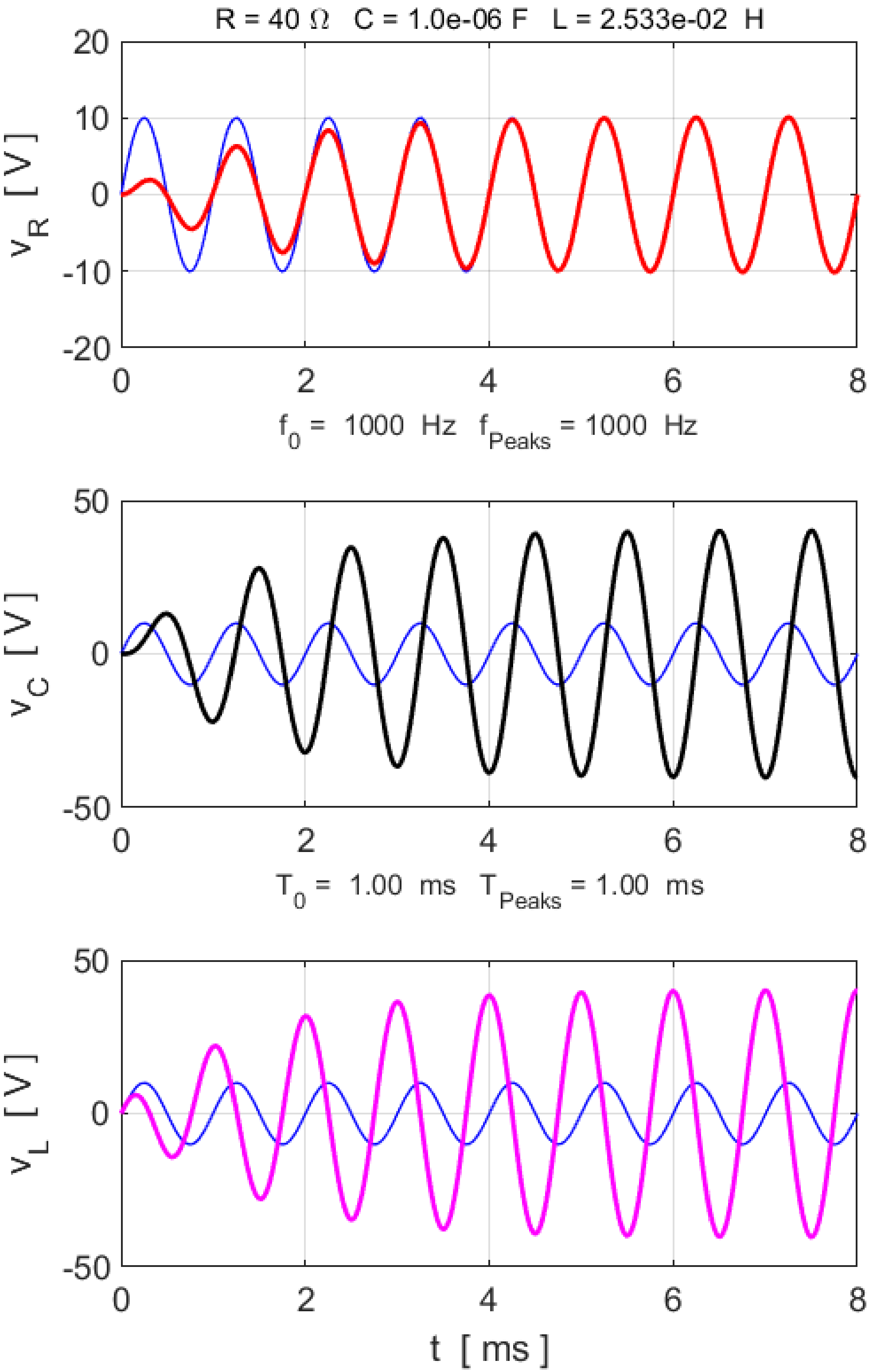
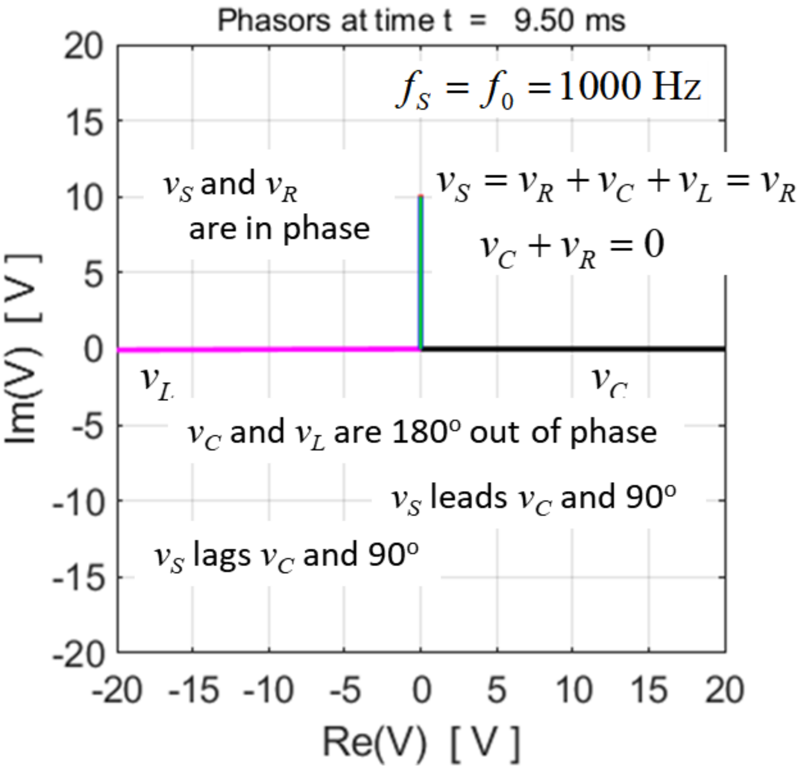


Fig. 12. The voltages as a function of time when the source frequency is equal to the natural frequency.

$$f_S = f_0 = 1000 \text{ Hz}$$



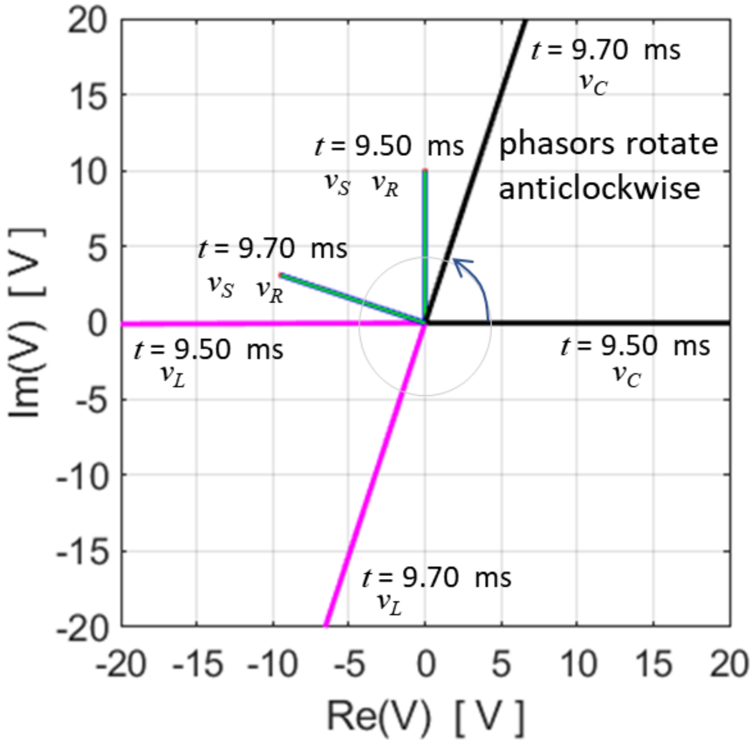


Fig. 13. Phasor diagram for the voltages at times t = 9.50 ms and time t = 9.70 ms. Each phasor rotates anticlockwise with angular velocity $\omega = 2\pi f$.

Remember, you cannot add ac voltages as simple numbers, they must be added like vector quantities. We can verify this in the Command Window by displaying the voltage and current values. You can compare the numerical results in the Table with the phasors in figure 13.

```
t(9700) = 0.0097

vS(9700) =  -9.4910 + 3.1499i

vR(9700) = -9.6073 + 3.1895i

vC(9700) = 12.6747 +38.2413i

vL(9700) = -12.5583 -38.2810i

vR(9700)+vC(9700)+vL(9700) =-9.4910 + 3.1499i

iS(9700) = -0.2402 + 0.0797i

rad2deg(angle(vS(9700))) =  161.6400

rad2deg(angle(vR(9700))) = 161.6343

rad2deg(angle(vC(9700))) = 71.6628

rad2deg(angle(vL(9700))) = -108.1624

rad2deg(angle(vL(9700))) = 161.6343

rad2deg(angle(iR(9700))) = 161.6343
```

The circuit impedances are calculated within the script and displayed in the Command Window.

```
Sinusoidal Source emf
  Source frequency  fS  =  1.00e+03  Hz
  Impedance: resistance   ZR  =  4.00e+01  ohms
  Impedance: capacitance   ZC  =  0.000e+00 -1.592e+02 ohms
    capacitance phase angle    phiC  =  -90.0 deg
  Impedance: inductance   ZL  =  0.000e+00 1.592e+02 ohms
    inductance phase angle    phiL   =  90.0 deg
  Impedance: total       Z   =  4.00e+01 -2.81e-12 ohms
  Impedance: magnitude    |Z| =  4.00e+01   ohms
    impedance phase angle phiZ     =  -0.00 deg
```

The impedance of the capacitor is equal in magnitude of the impedance of the inductor and 180^{o} out of phase at the resonance frequency. So, the total impedance is equal to the resistance value and the phase of the circuit impedance is zero.

The impedances given in the above Table were calculated from the equations

$$(5) \qquad \begin{array}{l} Z_R = R \quad Z_C = -j / \omega C \quad Z_L = j\,\omega C \\ Z = Z_R + Z_C + Z_L \end{array}$$

The impedance is defined as the ratio of the voltage to the current.

(6) $$Z = \left|\frac{v(t)}{i(t)}\right|$$ ***Z is independent of time***

We can calculate the values of the impedances in the Command Window using equation 6 at any time step and compare the values with the values given from relationships given in equation 5.

	t(9700) = 0.0097
Z_R	abs(vR(9700)/iS(9700) = 40.0000
Z_C	abs(vC(end)/iS(end)) = 159.1907
Z_L	abs(vL(end)/iS(end)) = 159.1956
Z	abs(vS(end)/iS(end)) = 39.5141

Figure 14 shows the phase plots (I vs V plot). The phase plot for the resistor is a straight line. The straight line indicates that the voltage and current for the resistor are in phase. The reciprocal of the slope of the line is equal to the value of the resistance.

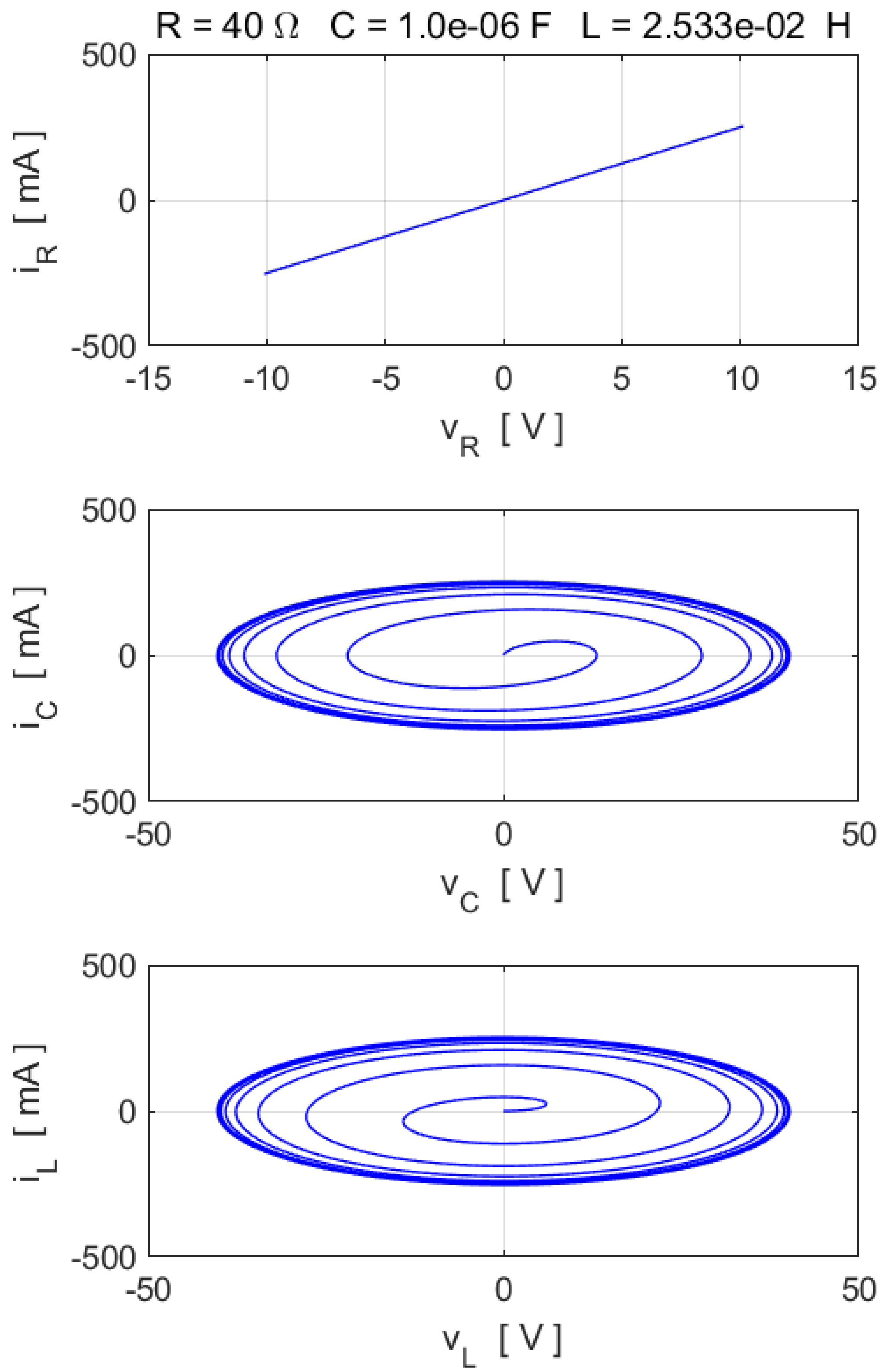


Fig .14. Phase plots showing how the current and voltage change with time.

The phase plots for the capacitor and inductor show that there is a 90° ($\pi / 2$ rad) phase difference between the voltage and current. The phase plots are ellipses – when the currents are zero, the voltages are a maximum and when the voltages are zero, the current are a maximum. For the capacitor phase plot, the path of the curve evolves in a clockwise direction which implies that the voltage lags the current. However, in the inductor phase plot the path of the curve evolves in an anticlockwise direction which implies that the voltage leads the current.

Figures 15 and 16 show the power and energy absorbed by the circuit elements plots as functions of time. The power $p_R(t)$ absorbed by the resistance is always positive which means that energy is dissipated by the resistance as thermal energy. For the capacitor and inductor, the time average powers $p_C(t)$ and $p_L(t)$ are both equal to zero. No energy is dissipated in our ideal capacitor or inductor. Energy is stored by the capacitor or inductor when the instantaneous power is positive and returned to the circuit when the instantaneous power is negative.

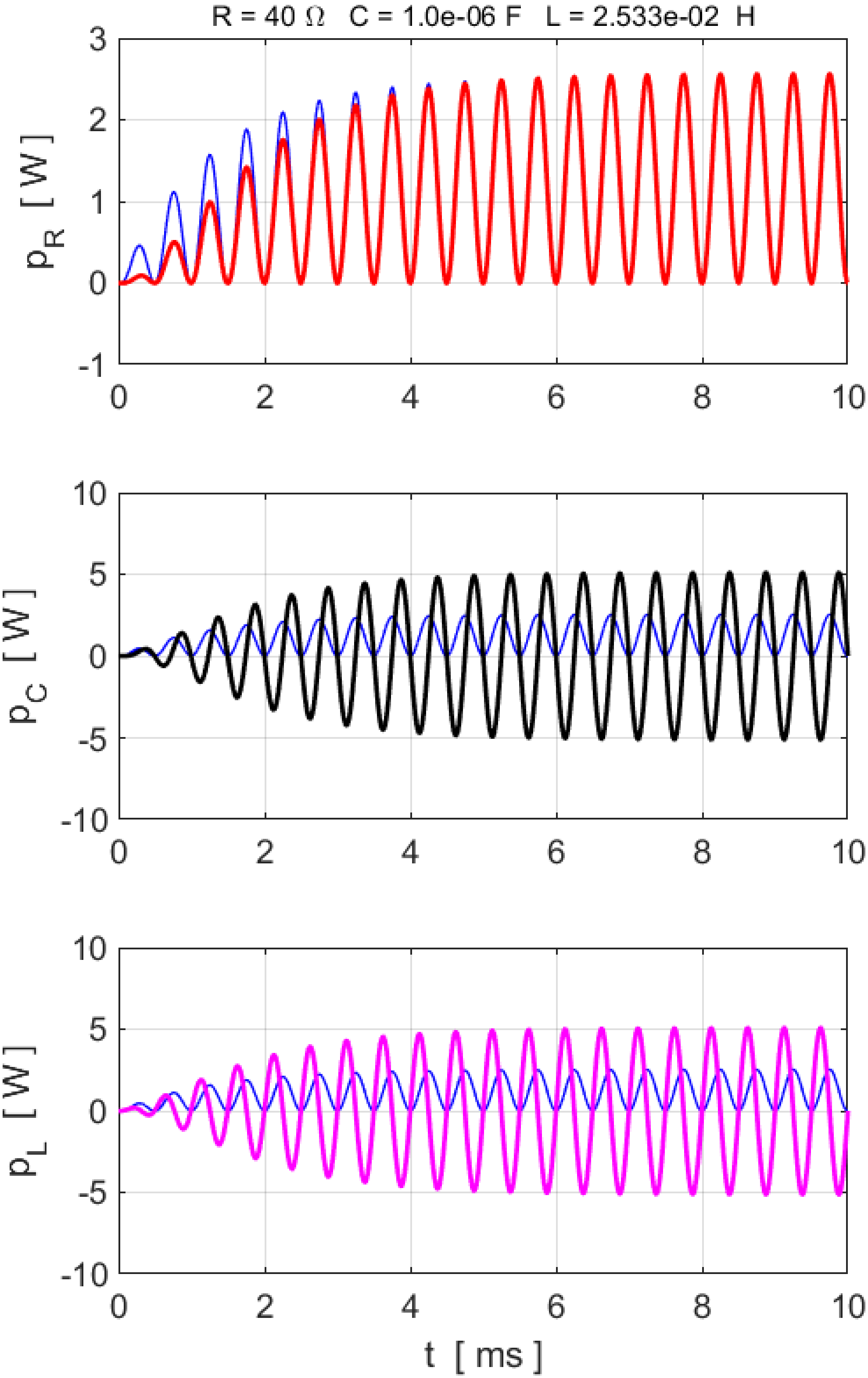


Fig. 15.   Power absorbed or supplied in the series RCL circuit at resonance $f_S = f_0 = 1000\ \mathrm{Hz}$.

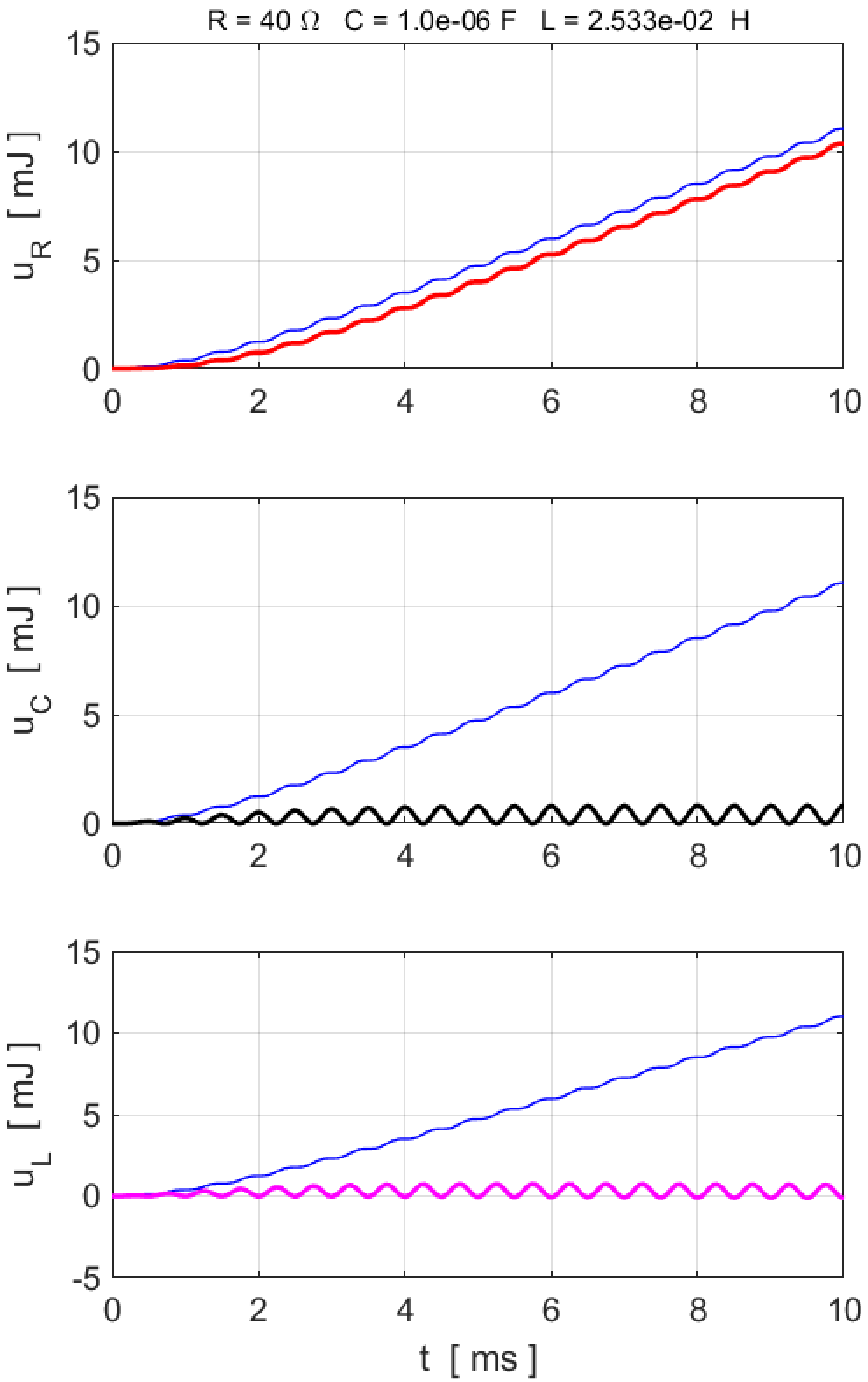


Fig. 16.   Energy absorbed or supplied in the series RCL circuit at resonance $f_S = f_0 = 1000\ \mathrm{Hz}$.

We can examine the response of the circuit for a sinusoidal source emf at a frequency above the resonance frequency $f_S = 1500 \text{ Hz} > f_0 = 1000 \text{ Hz}$.

Comparing figures 12 and 17 for the voltages as functions of time, there is a greater voltage across each element when the source frequency is equal to the resonance frequency. At resonance there is maximum current in the circuit (figure 8).

Element	$f_S = f_0 = 1000 \text{ Hz}$	$f_S = 1500 \text{ Hz}$
resistor V_{Rpeak}	10.0 V	2.9 V
capacitor V_{Cpeak}	40.3 V	7.7 V
inductor V_{Lpeak}	40.3 V	17.2 V

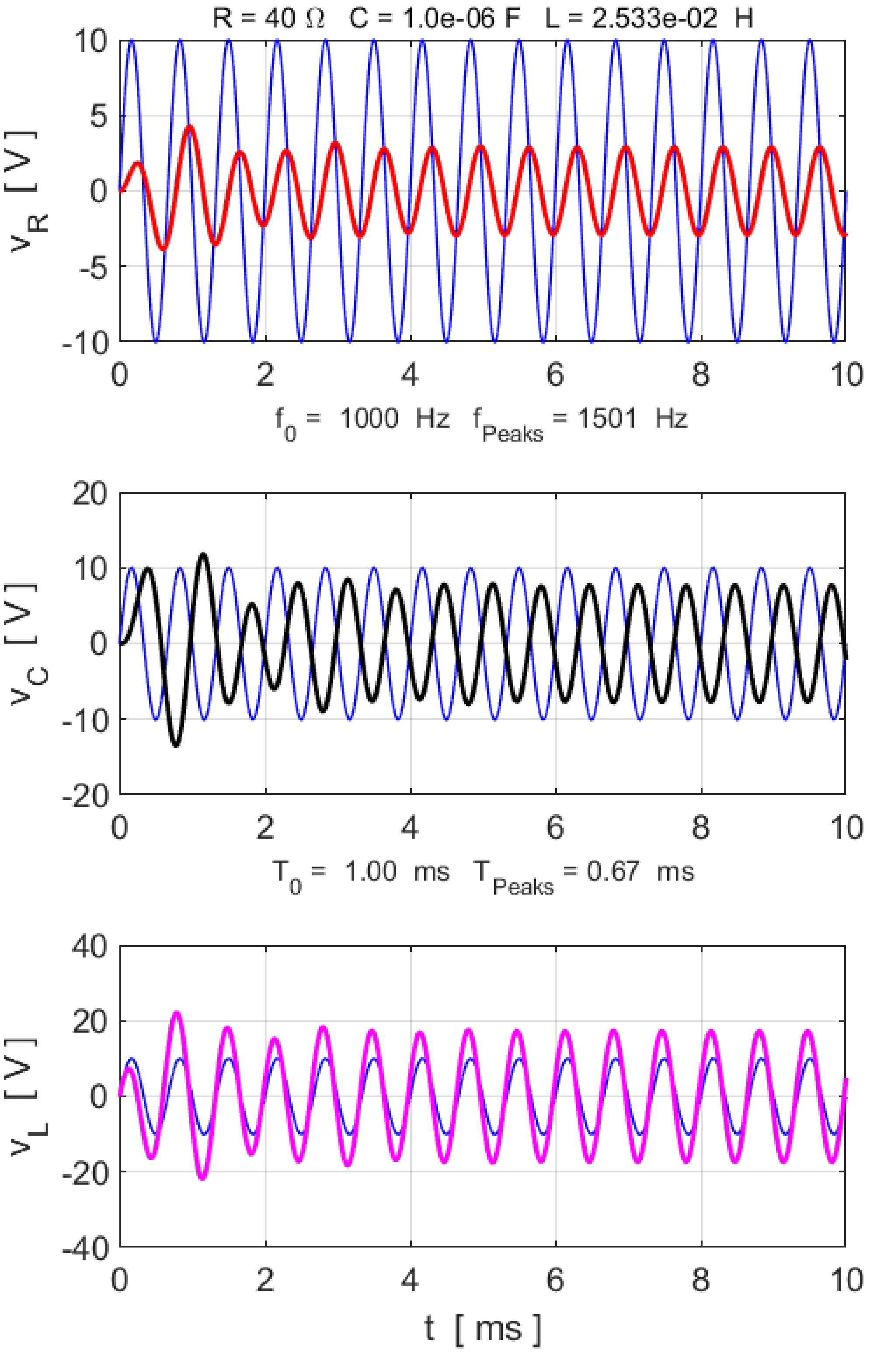


Fig. 17. The voltages as a function of time when the source frequency is greater than the natural frequency.

$f_S = 1500\ \text{Hz} > f_0 = 1000\ \text{Hz}$

The voltage across the resistor and the current through it are out of phase when the source frequency is not equal to the resonance frequency. Figure 18 shows the phasor diagram at time t = 9.50 ms. The capacitor and inductor voltages are still 180^{o} out of phase but the voltages have different magnitudes and so the effects of the capacitor and inductor do not cancel.

Phases [degrees] at time t = 9.50 ms
phiS = -0 deg phiR = -74 deg
phiC = -164 deg phiL = 17 deg
Phase difference between source emf & current
thetaSR = 74 deg

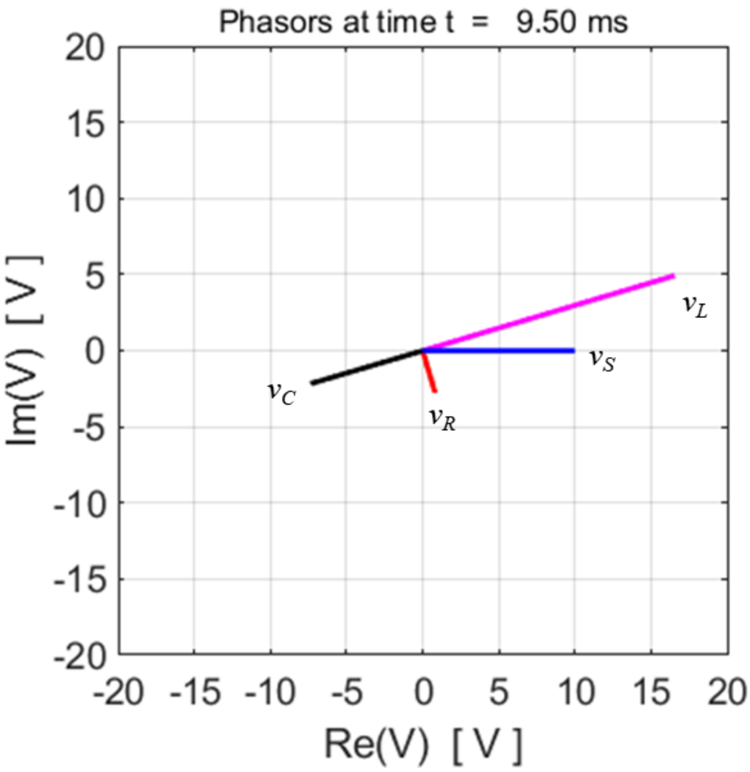


Fig. 18. Phasor diagram for the voltages at time t = 9.50 ms. The source emf leads the current by 74^{o}.