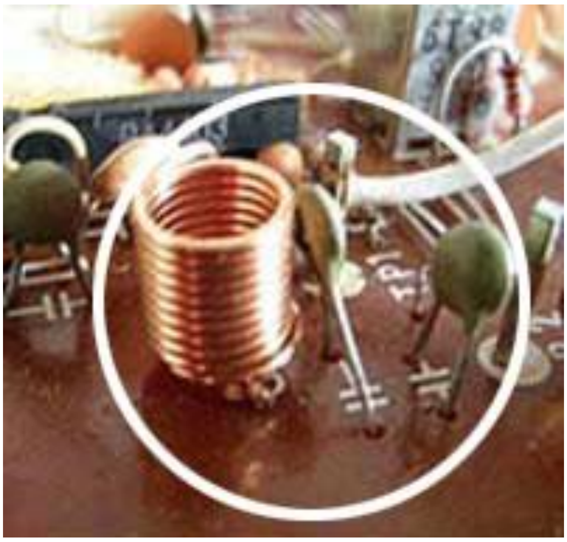
DOING PHYSICS WITH MATLAB

RESONANCE CIRCUITS

RLC PARALLEL VOLTAGE DIVIDER



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Matlab Download Directory

CRLCp1.m **CRLCp2.m**

When you change channels on your television set, an RLC circuit is used to select the required frequency. To watch only one channel, the circuit must respond only to a narrow frequency range (or frequency band) centred around the desired one. Many combinations of resistors, capacitors and inductors can achieve this. We will consider a RLC voltage divider circuit shown in figure 1. The circuit shown in figure 1 can also be used as a narrow band pass filter or an oscillator circuit.

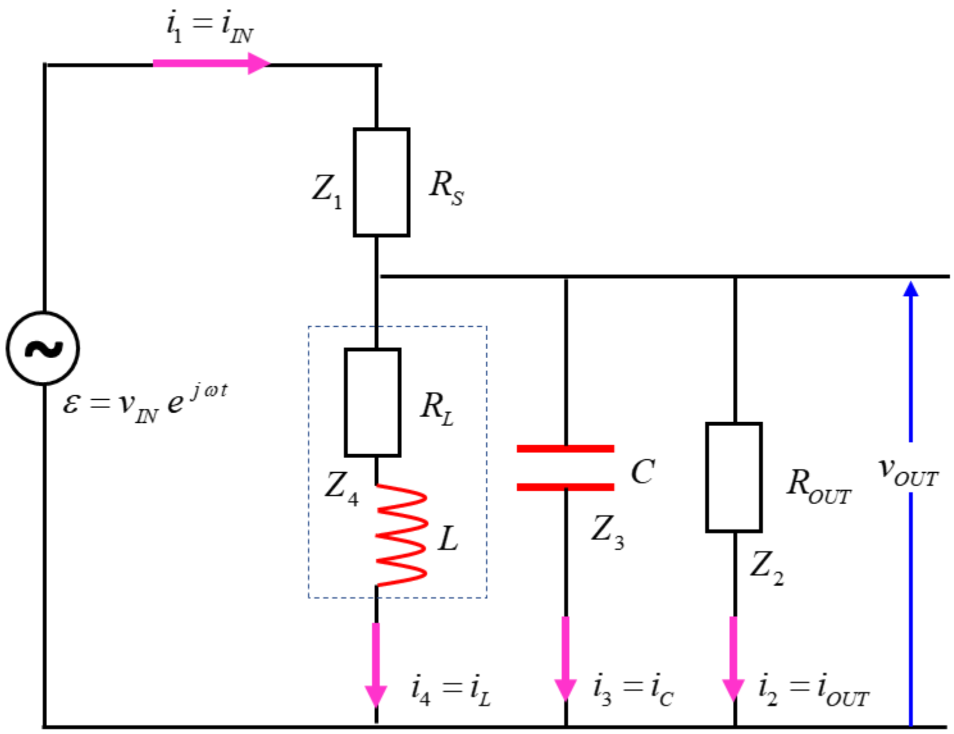

Fig. 1. RLC resonance circuit: a parallel combination of an inductor L and a capacitor C used in voltage divider circuit.

The sinusoidal input voltage is

$$\varepsilon = V_{IN}\, \mathrm{e}^{\,j\omega t}$$

The impedances of the circuits components are

$Z_1 = R_1$ series resistance

$Z_2 = R_{OUT}$ output or load resistance

$Z_3 = \dfrac{-j}{\omega C}$ capacitor

$Z_4 = R_L + j\omega L$ inductor

We simplify the circuit by combining circuit elements that are in series and parallel.

$Z_5 = \dfrac{1}{\dfrac{1}{Z_2} + \dfrac{1}{Z_3} + \dfrac{1}{Z_4}}$ parallel combination

$Z_6 = Z_1 + Z_5$ series combination: total impedance

The current through each component and the potential difference across each component is computed from

$$I = \frac{V}{Z} \qquad V = I\,Z$$

in the following sequence of calculations

$$I_1 = \frac{V_{IN}}{Z_6}$$

$$V_1 = I_1 Z_1$$

$$V_{OUT} = V_{IN} - V_1$$

$$I_2 = \frac{V_{OUT}}{Z_2} \quad I_3 = \frac{V_{OUT}}{Z_3} \quad I_4 = \frac{V_{OUT}}{Z_4}$$

Computing all the numerical values is easy using the complex number commands in Matlab. Complex circuits can be analysed and in more in-depth graphically than the traditional algebraic approach.

The code below shows the main calculations that needed for the simulations.

```
    f = linspace(fMin,fMax, N);
    w = (2*pi).*f;

% impedances
    Z1 = RS;                          % series
resistance
    Z2 = ROUT;                        % output or load
resistance
    Z3 = -1i ./ (w .*C);              % capacitive
impedance
    Z4 = RL + 1i .* w .* L;           % inductive
impedance
                                          (resistance +
reactance)

    Z5 = 1./ (1./Z2 + 1./Z3 + 1./Z4);    % parallel
combination
    Z6 = Z1 + Z5;                        % total
circuit

impedance

% currents [A] and voltages [V]
    I1 = V_IN ./ Z6;
    V1 = I1 .* Z1;
    V_OUT = V_IN - V1;

    I2 = V_OUT ./ Z2;
    I3 = V_OUT ./ Z3;
    I4 = V_OUT ./ Z4;
    I_sum = abs(I1 - I2 - I3 - I4);

% phases
    phi_OUT = angle(V_OUT);
    phi_1   = angle(V1);

    theta_1 = angle(I1);
    theta_2 = angle(I2);
    theta_3 = angle(I3);
```

```
    theta_4 = angle(I4);

% Resonance frequencies and Bandwidth calculations
    f0 = 1/(2*pi*sqrt(L*C));
    G_V = abs(V_OUT ./ V_IN);        % voltage gain
    Vpeak = max(G_V);                % max voltage
gain
    VG3dB = Vpeak/sqrt(2);           % 3 dB points
    k = find(G_V == Vpeak);          % index for peak
voltage gain
    f_peak = f(k);                   % frequency at
peak
    kB = find(G_V > VG3dB);          % indices for 3dB
peak
    k1 = min(kB); f1 = f(k1);
    k2 = max(kB); f2 = f(k2);
    df = f2-f1;                      % bandwidth
    Q = f0 / df;                     % quality factor

    P_OUT = V_OUT .* I2;             % power delivered
to load
```

We will consider the following example that was done as an experiment and as a computer simulation.

Values for circuit parameters:

amplitude of input emf	$V_{in} = 10.0\,\text{V}$
series resistance	$R_S = 1.00 \times 10^4\ \Omega$
capacitance	$C = 10.4 \times 10^{-9}\,\text{F} \quad (10\ \text{nF})$
inductance	$L = 10.3 \times 10^{-3}\,\text{H} \quad (10\ \text{mH})$
inductance resistance	estimated from simulation
output (load) resistance	$R_{OUT} = 1.00 \times 10^6\ \Omega$ (output to CRO)

Simulation $R_L = 0\,\Omega$ **script CRLCp1.m**

```
%   INPUTS   default values [ ]
% series resistance Z1 [1e4 ohms]
   RS = 1e4;
% OUTPUT (LOAD) resistance Z2 [1e6 ohms]
   ROUT = 1e6;
% inductance and inductor resistance Z4
[10.3e-3 H  0 ohms]
   L = 10.3e-3;
   RL = 0;
% capacitance Z2 [10.4e-9 F]
   C = 10.4e-9;
% input voltage emf [10 V]
   V_IN = 10;
% frequency range [1000 to 30e3 Hz   5000]
   fMin = 1000; fMax = 30e3;
   N = 5000;
```

Figure 2 shows the plots of the absolute values for the impedance of the capacitor (Z_3), inductor (Z_4) and output impedance for the parallel combination (Z_5).

At low frequencies, the inductor acts like a “short circuit”

$$Z_L \equiv Z_4 = j\omega L$$
$$f \to 0 \quad \Rightarrow \quad Z_4 \to 0 \quad \Rightarrow \quad Z_{out} \to 0$$

At high frequencies, the capacitor acts like a “short circuit”

$$Z_C \equiv Z_3 = \frac{-j}{\omega C}$$
$$f \to 0 \quad \Rightarrow \quad Z_3 \to 0 \quad \Rightarrow \quad Z_{out} \to 0$$

At resonance $|Z_L| = |Z_C|$

$$\omega L = \frac{1}{\omega C}$$

resonance frequency $\omega_0 = \dfrac{1}{\sqrt{LC}} \quad f_0 = \dfrac{1}{2\pi\sqrt{LC}}$

The output impedance $Z_{out} \equiv Z_5$ has a sharp peak at the resonance frequency f_0.

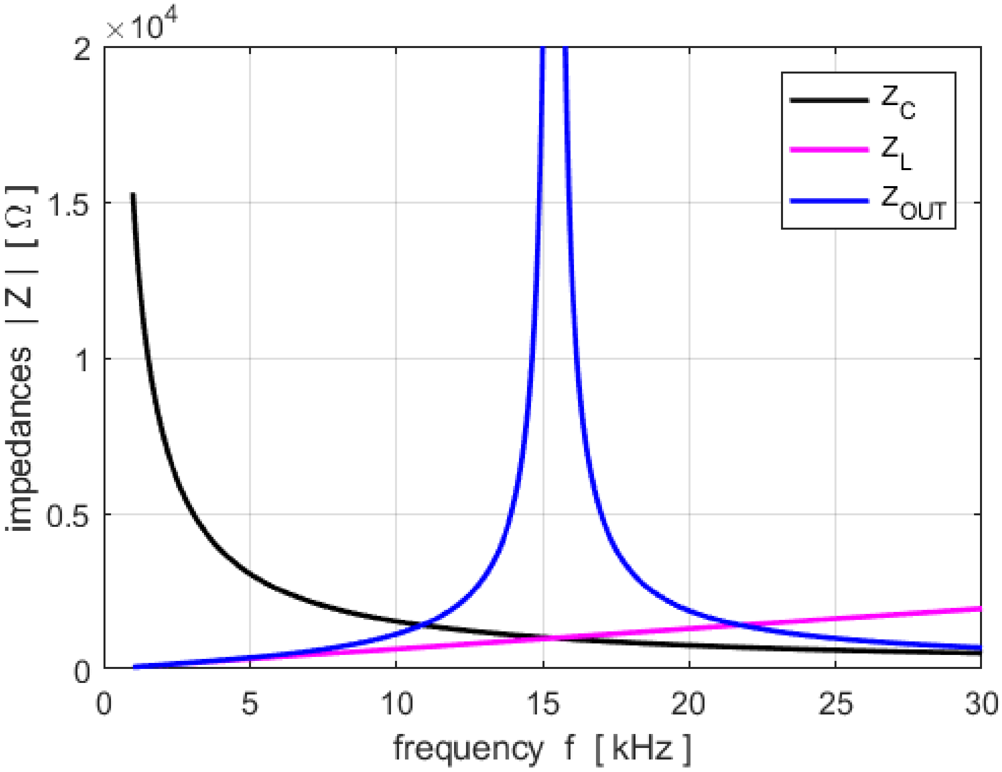

Fig. 2. The magnitude of the impedances for the capacitor, inductor and parallel combination as functions of frequency of the source. A sharp peak occurs at the resonance frequency for the impedance of the parallel combination.

Since the total circuit impedance has a maximum value at the resonance, the current from the source must be a minimum at the resonance frequency (figure 3). At resonance, the source voltage and the source current are in-phase with each other and the current through the series resistance is the same as the current through the output resistance.

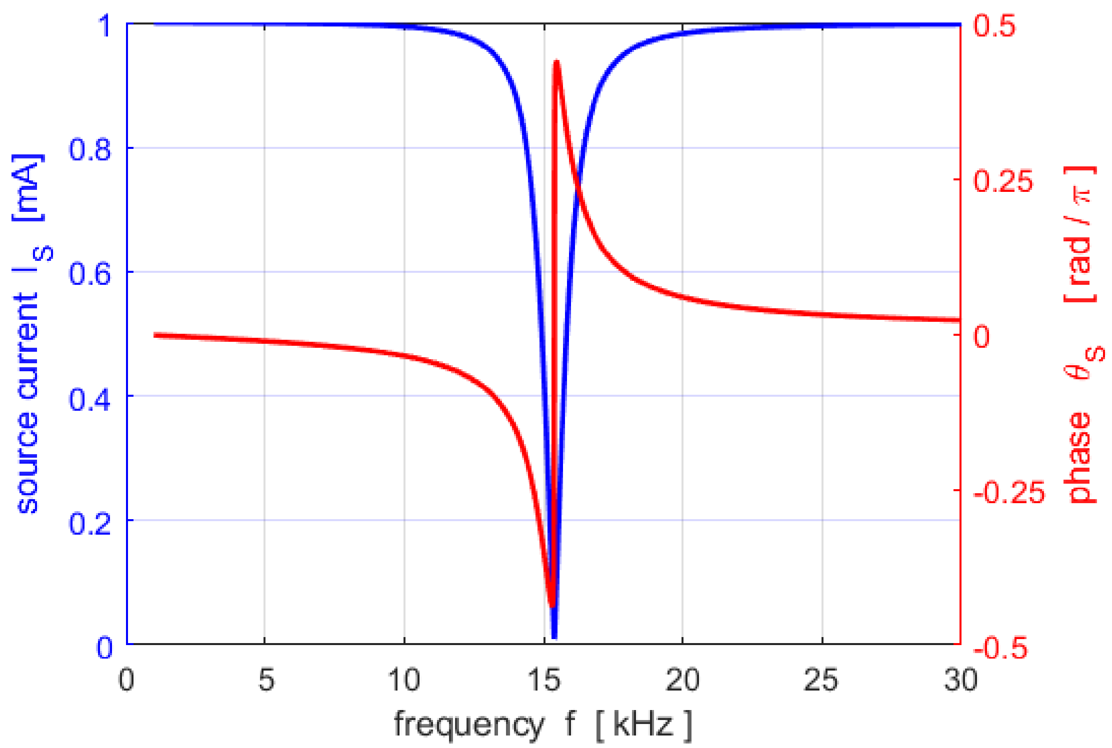

Fig. 3. The source current has a minimum at the resonance frequency. At resonance, the source voltage and source current are in-phase with each other.

The output voltage from the parallel combination is

$$v_{OUT} = V_{OUT}\, e^{\,j\omega t + \phi_{OUT}}$$

We define the voltage gain G_V of the circuit as

$$G_V = \left| \frac{V_{OUT}}{V_{IN}} \right|$$

The G_V is a complex quantity which is specified by its magnitude and phase. Figure 4, shows the magnitude of the voltage gain G_V and its phase ϕ_{OUT}. The voltage gain G_V has a peak at the resonance frequency and the output voltage v_{OUT} is in phase with the source voltage ε since $\phi_{OUT} = 0$.

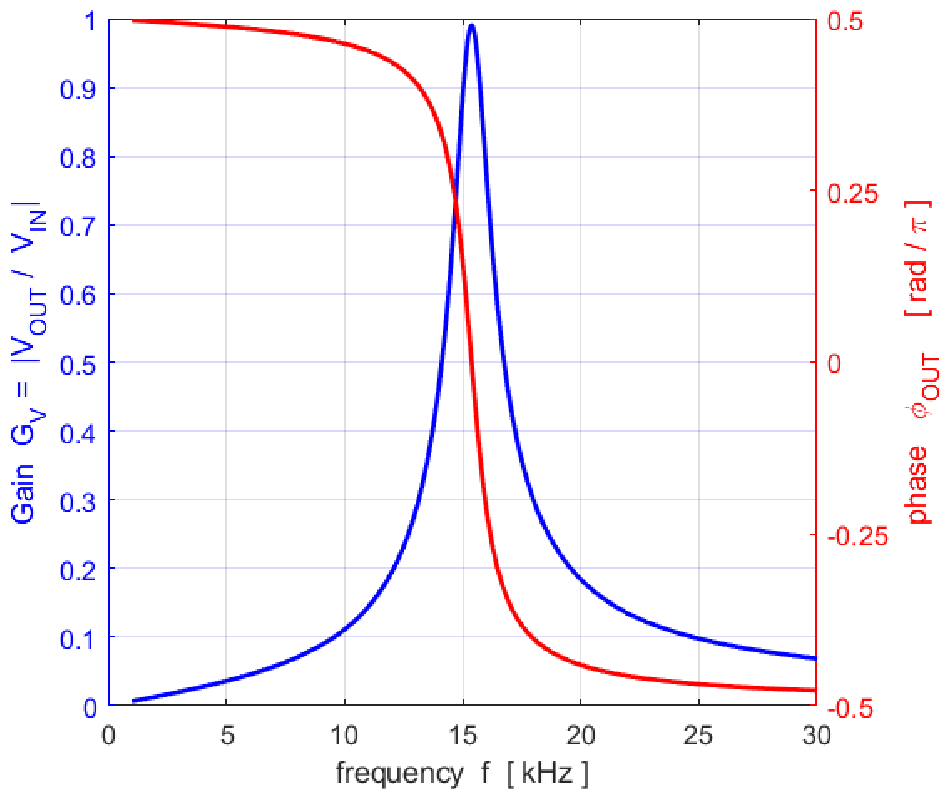

Fig. 4. The voltage gain G_V of the parallel voltage divider resonance circuit.

For the case when $R_L = 0\,\Omega$, the **resonance frequency** of the circuit is

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

The **quality factor** Q is a measure of the width of the voltage gain plot. The power drops by half (-3 dB) at the half power frequencies f_1 and f_2 where $G_V = 1/\sqrt{2}$. These two frequencies determine the **bandwidth** Δf of the voltage gain peak

$$\Delta f = f_2 - f_1$$

It can be shown that the quality factor Q is

$$Q = \frac{f_0}{\Delta f}$$

The higher the Q value of a resonance circuit, the narrow the bandwidth and hence the better the selectivity of the tuning.

Figure 5 shows the voltage gain plot indicating the resonance frequency, half power frequencies and the bandwidth.

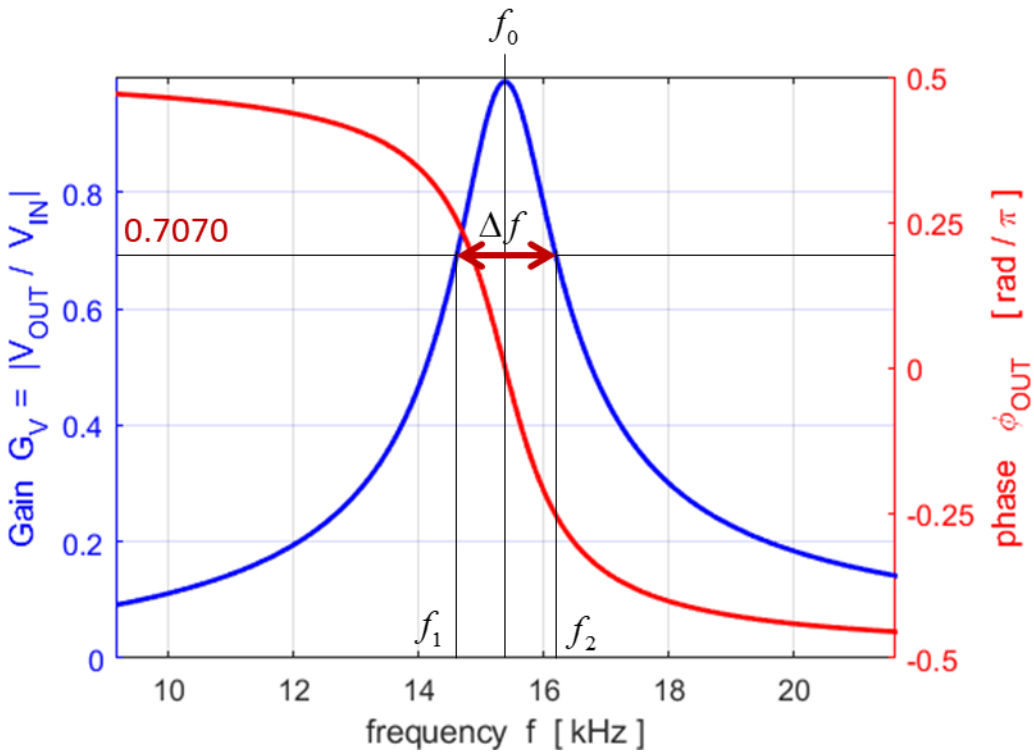

Fig. 5. The voltage gain plot indicating the resonance frequency, half power frequencies and the bandwidth.

A summary of the calculations is displayed in the Command Window

Resonance frequency	$f_0 = 15377\ \text{Hz}$
Peak frequency	$f_{peak} = 15375\ \text{Hz}$
Half power frequencies	$f_1 = 14627\ \text{Hz}$ $\quad f_2 = 16164\ \text{Hz}$
Quality factor	$Q = 10$

```
%  OUTPUTS IN COMMAND WINDOW
  fprintf('theoretical resonance frequency
f0 = %3.0f Hz \n',f0);
  fprintf('peak frequency f_peak = %3.0f Hz
\n',f_peak);
  fprintf('half power frequencies f1 = %3.0f
Hz  %3.0f  Hz \n',f1,f2);
  fprintf('bandwidth   df = %3.0f Hz \n',df);
  fprintf('quality factor  Q  =  %3.2f
\n',Q);
  fprintf('current at junction I_sum
=  %3.2f mA  \n',max(1e3*I_sum))
```

Figure 6 shows the magnitudes of the currents through the capacitor and inductor branches of the parallel combination and the corresponding phases.

For frequencies less than the resonance frequency, current through the inductive branch is greater than through the capacitive branch. At resonance, the two currents are equal. Above the resonance frequency, there is more current in the capacitive branch than the inductive branch.

The two currents are always π rad out of phase. The capacitive current leads by $\pi / 2 \text{ rad}$ while the inductive current lags by $\pi / 2 \text{ rad}$, compared with the reference angle of the source emf.

At resonance, $\theta_C = +\pi / 2 \text{ rad} \quad \theta_L = -\pi / 2 \text{ rad}$ and the two currents have the same magnitudes. Therefore, the effects of the capacitance and inductance cancel each other, resulting in a pure resistive impedance with the source voltage and current in phase.

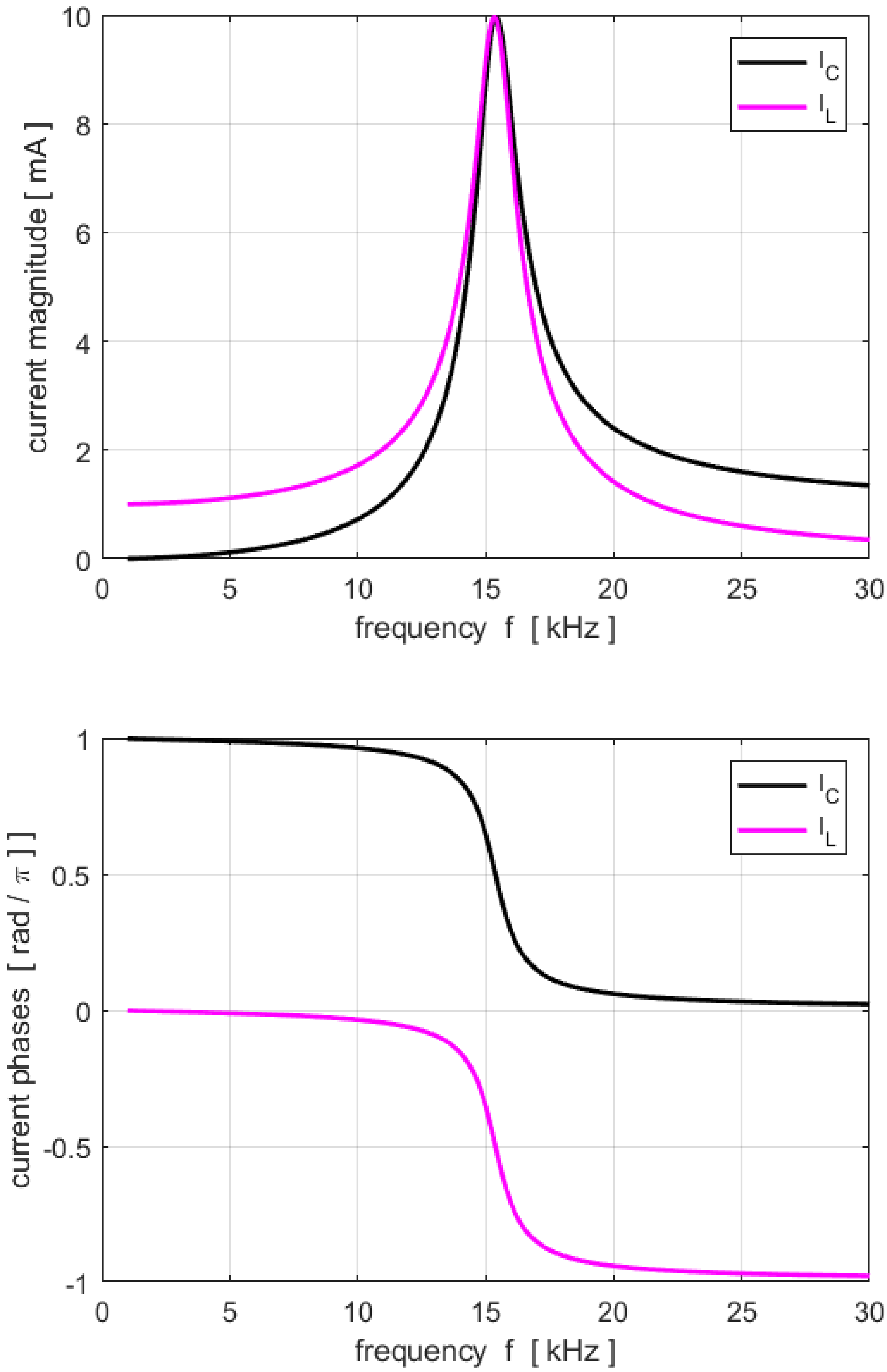

Fig. 6. Magnitudes and phases for the inductor current and inductor current.

Kirchhoff's current law states that the sum of the currents at a junction add to zero. For ac circuits, it is not so straight forward to sum the currents because you must account for the phases of each current. At the junction of the series resistance and the parallel combination, the simulation gives the result

$\sum \vec{I} = -\vec{I}_1 + \vec{I}_2 + \vec{I}_3 + \vec{I}_4 = 0$ $\rightarrow$ need to account for phase

script: `I_sum = abs(I1 - I2 - I3 - I4);`

output: `current at junction I_sum  =  0.00 mA`

Maximum power $(P = vi)$ is delivered from the source to the load only at the resonance frequency as shown in figure 7.

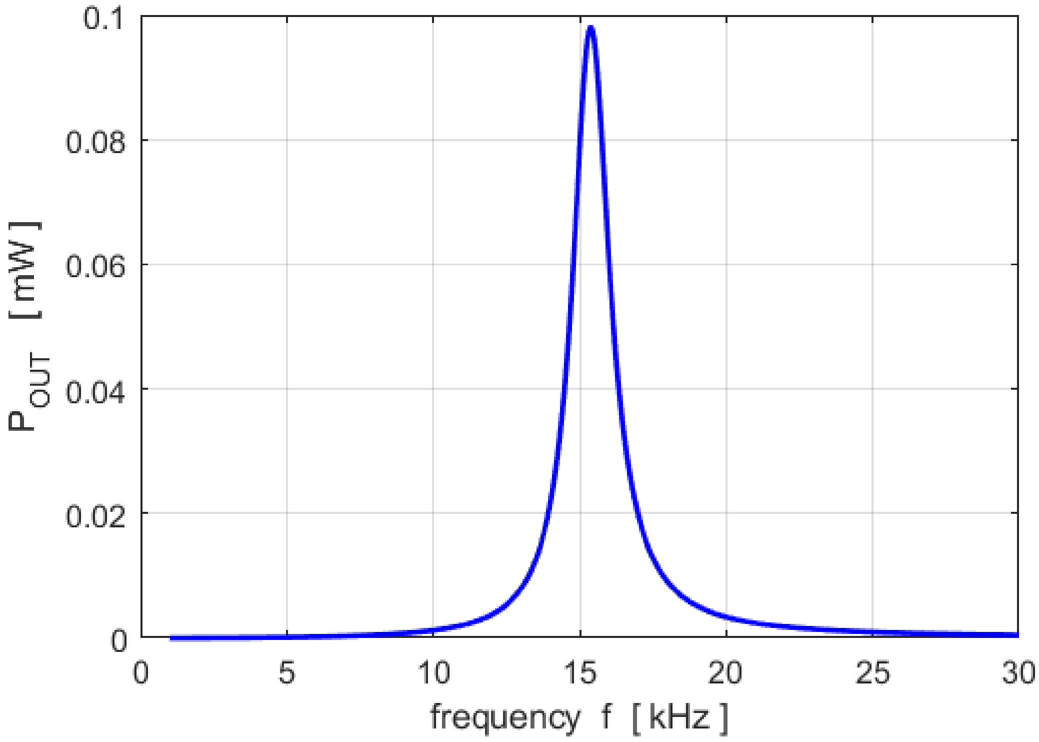

Fig. 7. Power delivered from source to load.

We can also look at the behaviour of the circuit in the **time domain** and gain a better understanding of how complex numbers give us information about magnitudes and phases. The time domain equation for the currents and voltages are

$$\varepsilon = V_{IN}\, \mathrm{e}^{\,j\omega t}$$

$$v_1 = V_1\, e^{\,j(\omega t + \phi_1)}$$

$$v_{OUT} = V_{OUT}\, e^{\,j(\omega t + \phi_{OUT})}$$

$$i_1 \equiv i_S = I_1\, e^{\,j(\omega t + \theta_1)}$$

$$i_2 \equiv i_{OUT} = I_2\, e^{\,j(\omega t + \theta_2)}$$

$$i_3 \equiv i_C = I_3\, e^{\,j(\omega t + \theta_3)}$$

$$i_4 \equiv i_L = I_4\, e^{\,j(\omega t + \theta_4)}$$

Each of the above relationship are plotted at a selected frequency which is set within the script. The graphs below are for the resonance frequency and the half-power frequencies.

```
% TIME DOMAIN    select source frequency
     fs = f_peak; kk = k;
   % fs = f1; kk = k1;
   % fs = f2; kk = k2;

   Ns = 500;
   ws = 2*pi*fs;
   Ts = 1/fs;
   tMin = 0;
   tMax = 3*Ts;
   t = linspace(tMin,tMax,Ns);

   emf = real(V_IN .* exp(1j*ws*t));
   v_OUT = real(abs(V_OUT(kk)) .*
exp(1j*(ws*t + phi_OUT(kk))));
   v1 = real(abs(V1(kk)) .* exp(1j*(ws*t +
phi_1(kk))));

   i1 = real(abs(I1(kk)) .* exp(1j*(ws*t +
theta_1(kk))));
   i2 = real(abs(I2(kk)) .* exp(1j*(ws*t +
theta_2(kk))));
   i3 = real(abs(I3(kk)) .* exp(1j*(ws*t +
theta_3(kk))));
   i4 = real(abs(I4(kk)) .* exp(1j*(ws*t +
theta_4(kk))));
```

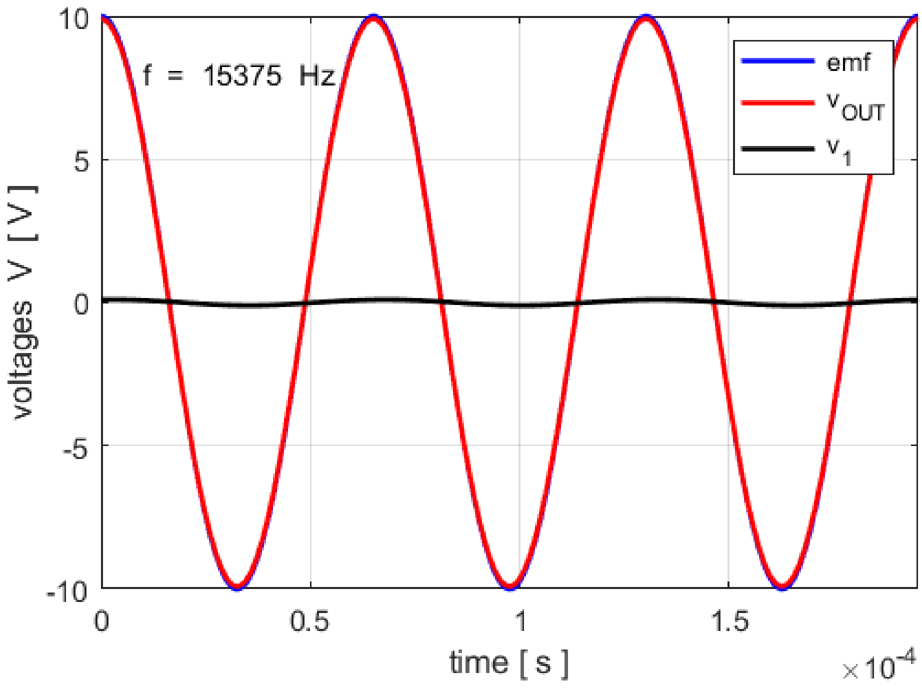

Fig. 8. The voltages at the resonance frequency. The source voltage and output voltage are in-phase. The output voltage is almost equal in magnitude to the source emf.

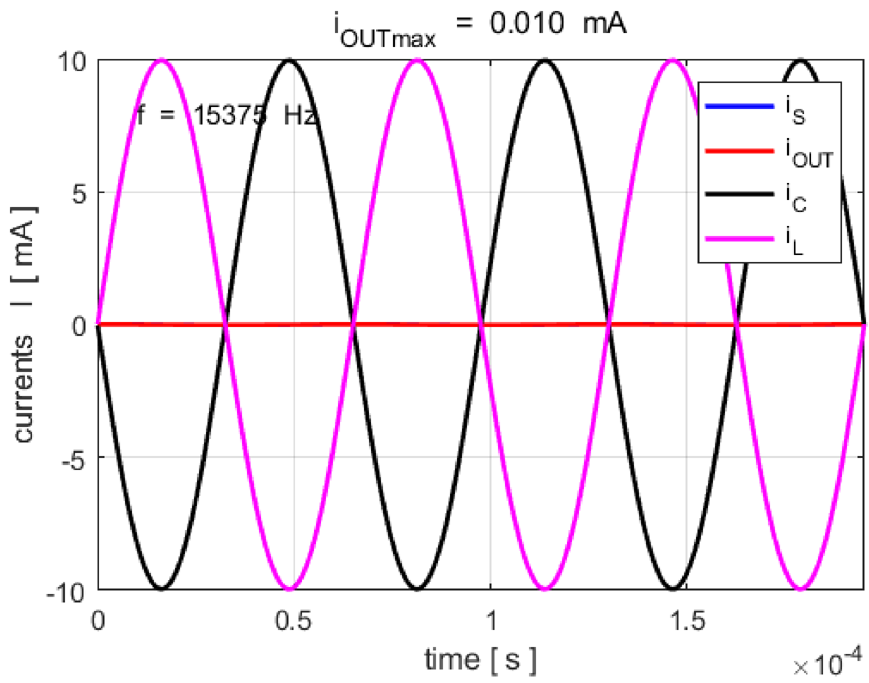

Fig. 9. The currents through the capacitor and inductor cancel at resonance. The currents through the capacitor and the inductor

are π rad out-of-phase and have equal amplitudes. The impedance of the circuit is purely resistive at the resonance frequency. The output current varies sinusoidally but has a very small amplitude.

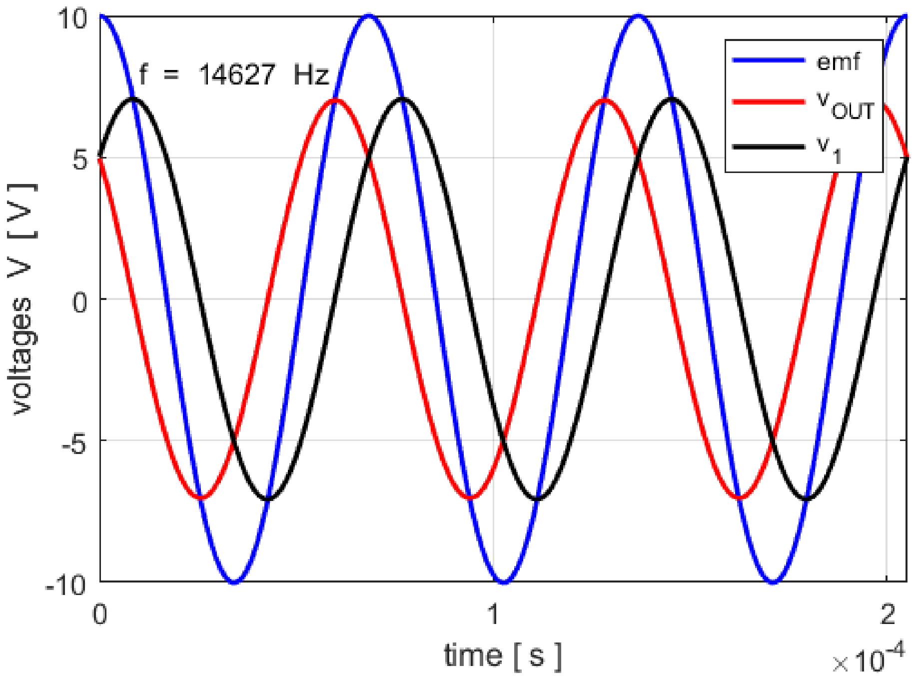

Fig. 10. The voltages at the half-power frequency f_1. The source voltage and output voltage are out-of-phase. At each instant $emf = v_1 + v_{OUT}$.

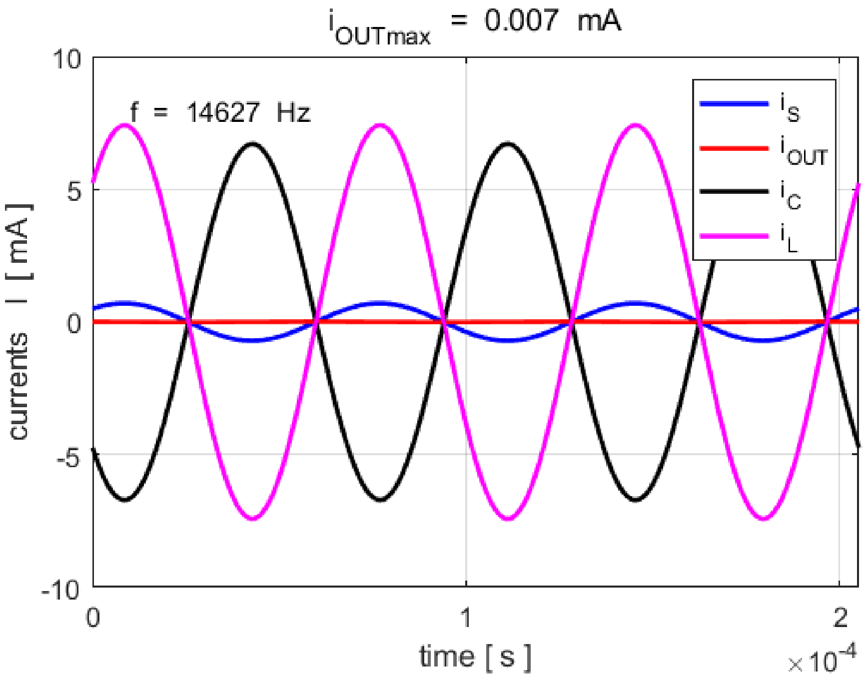

Fig. 11. The currents at the half-power frequency f_1. The currents through the capacitor and the inductor are π rad out-of-phase. The amplitude of the inductor current is greater than the amplitude of the capacitor current. The output current varies sinusoidally but has a very small amplitude. At each instant $i_S = i_{OUT} + i_C + i_L$

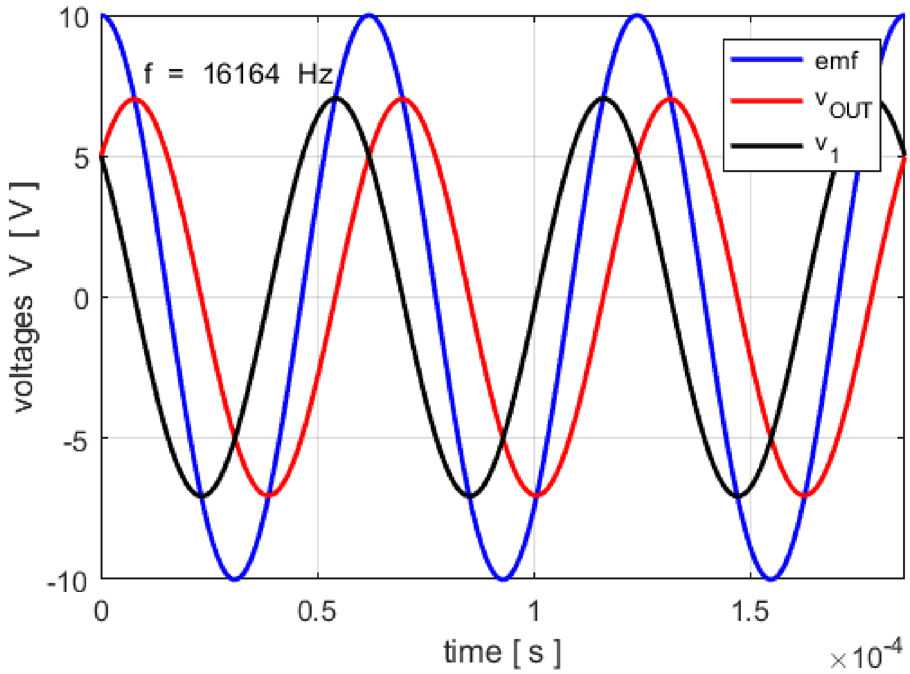

Fig. 12. The voltages at the half-power frequency f_2. The source voltage and output voltage are out-of-phase. At each instant $emf = v_1 + v_{OUT}$.

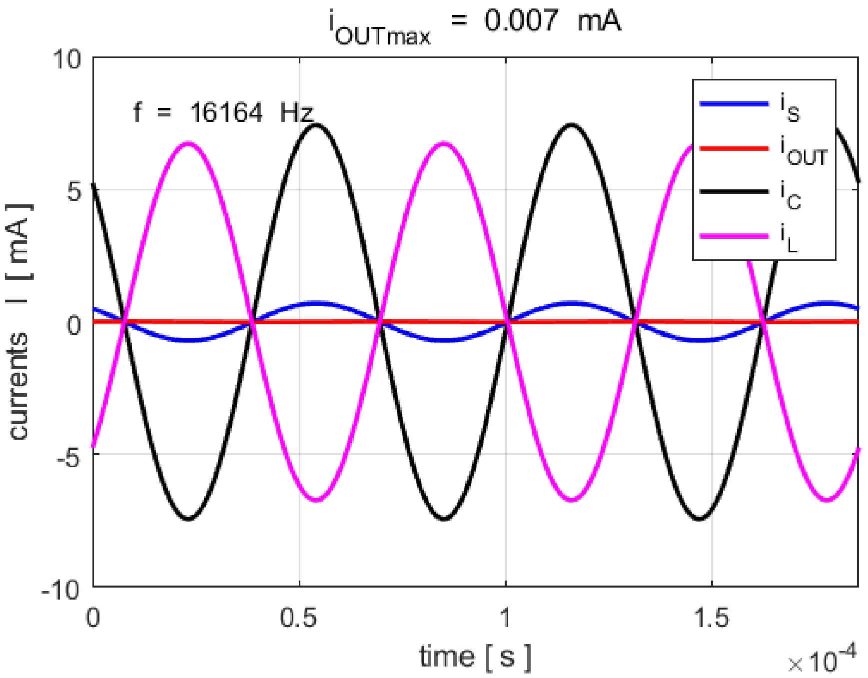

Fig. 13. The currents at the half-power frequency f_2. The currents through the capacitor and the inductor are π rad out-of-phase. The amplitude of the inductor current is less than the amplitude of the capacitor current. The output current varies sinusoidally but has a very small amplitude. At each instant $i_S = i_{OUT} + i_C + i_L$

The above simulation has for a large impedance load connected to the output of the circuit $R_{OUT} = 1.00 \times 10^6\ \Omega$. We can examine the effect when a much smaller load resistance is connected to the circuit while keeping all other parameters unchanged.

$$R_{OUT} = 1.00 \times 10^4\ \Omega$$

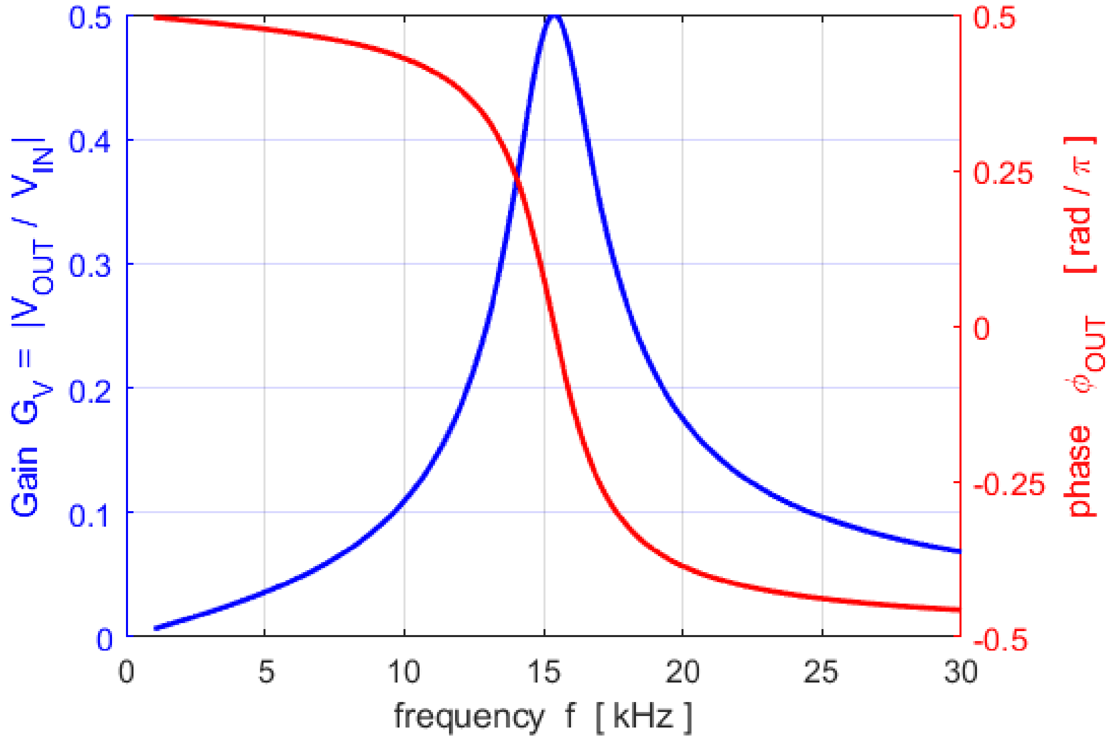

Fig. 14. The voltage gain is much reduced and the bandwidth increased giving a smaller quality factor Q. The larger bandwidth and smaller Q means that the selectivity is of the resonance circuit is reduced.

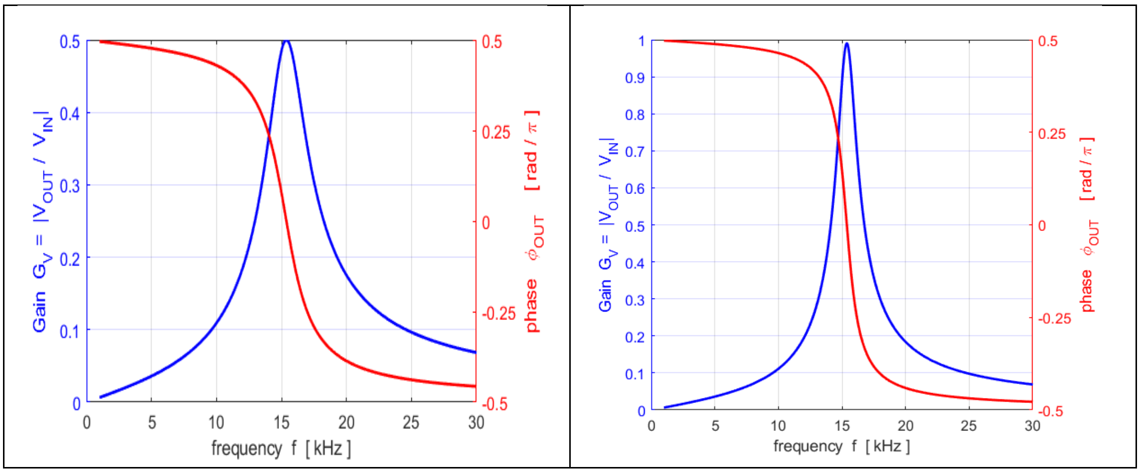

Fig. 15. Left $R_{OUT} = 1.00 \times 10^4\ \Omega$ Right $R_{OUT} = 1.00 \times 10^6\ \Omega$

Since the output resistance is reduced, a much greater output current results and more power is delivered from the source to the load as shown in figure 16. With $R_{OUT} = 1.00 \times 10^6\ \Omega$ the maximum power at resonance is 0.1 mW, whereas when $R_{OUT} = 1.00 \times 10^4\ \Omega$, the maximum power at resonance 2.5 mW.

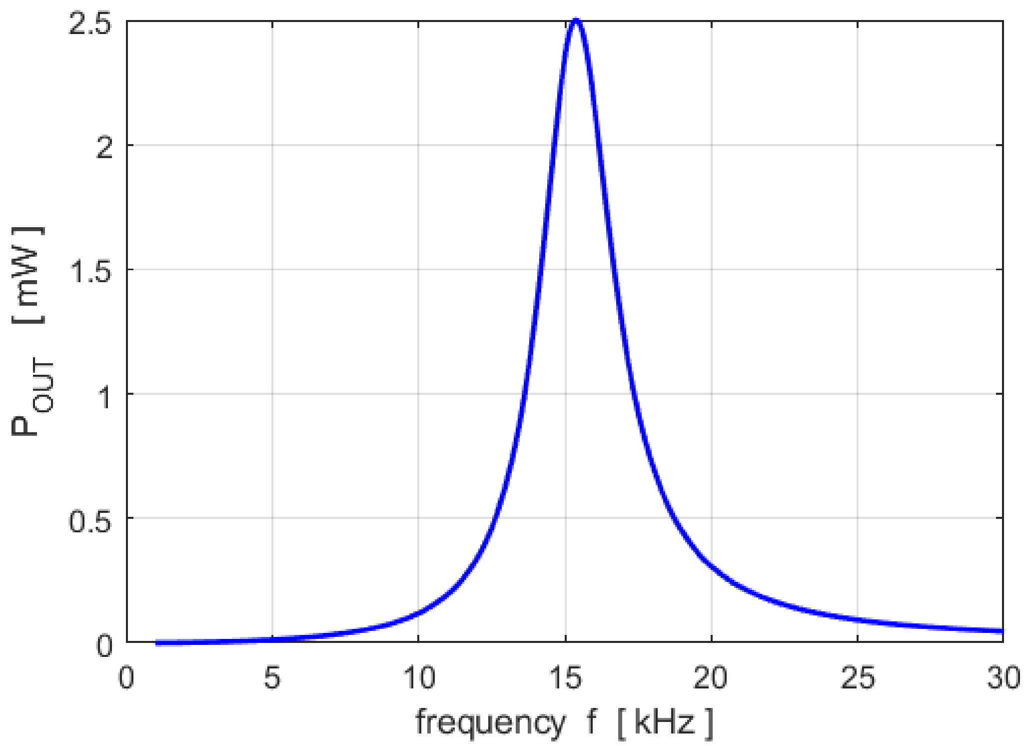

Fig. 16. The power delivered to the load is increased as the output resistance is decreased.

Modelling Experimental Data

Data was measured for the circuit shown in figure 1. An audio oscillator was used for the source and the output was connected to digital storage oscilloscope (DSO). The component values used were (nominal values shown in brackets):

series resistance	$R_S = 1.00 \times 10^4\ \Omega$
capacitance	$C = 10.4 \times 10^{-9}\ \mathrm{F} \quad (10\ \mathrm{nF})$
inductance	$L = 10.3 \times 10^{-3}\ \mathrm{H} \quad (10\ \mathrm{mH})$
assume DSO resistance	$R_{OUT} = 1.00 \times 10^6\ \Omega$ (output to CRO)

The measurements are given in the script CRLCp2.m

Figure 17 shows a plot of the experimental data.

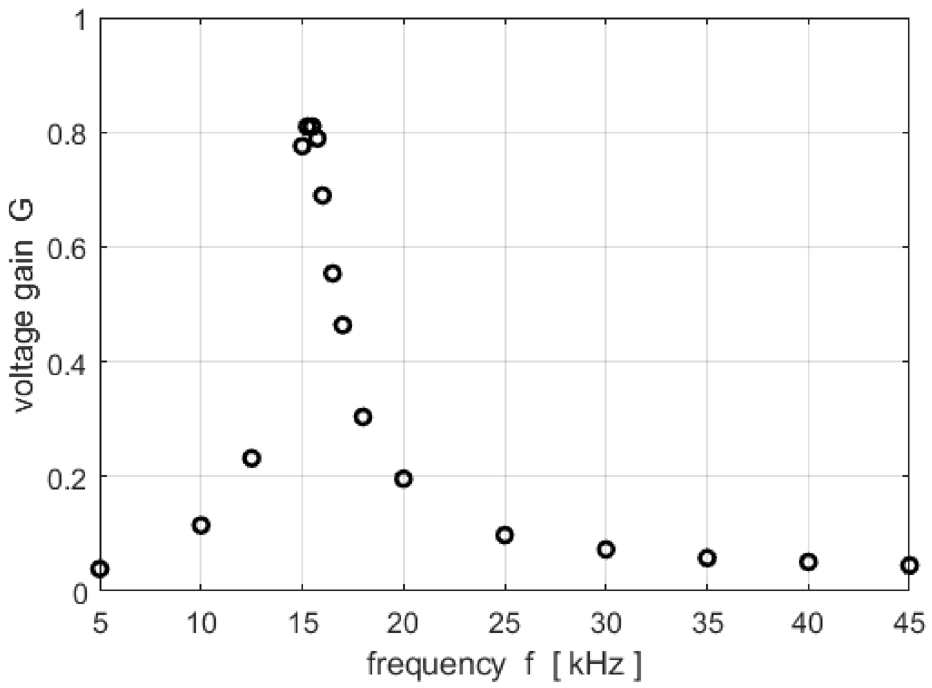

Fig. 17. Plot of the experimental measurements.

We can use our simulation **CRLCp2.m** to fit theoretical curves to the measurements and estimate the resistance of the inductor as shown in figures 18 and 19. The value of the inductive resistance R_L is changed to obtain the best fit between the model and the measurements.

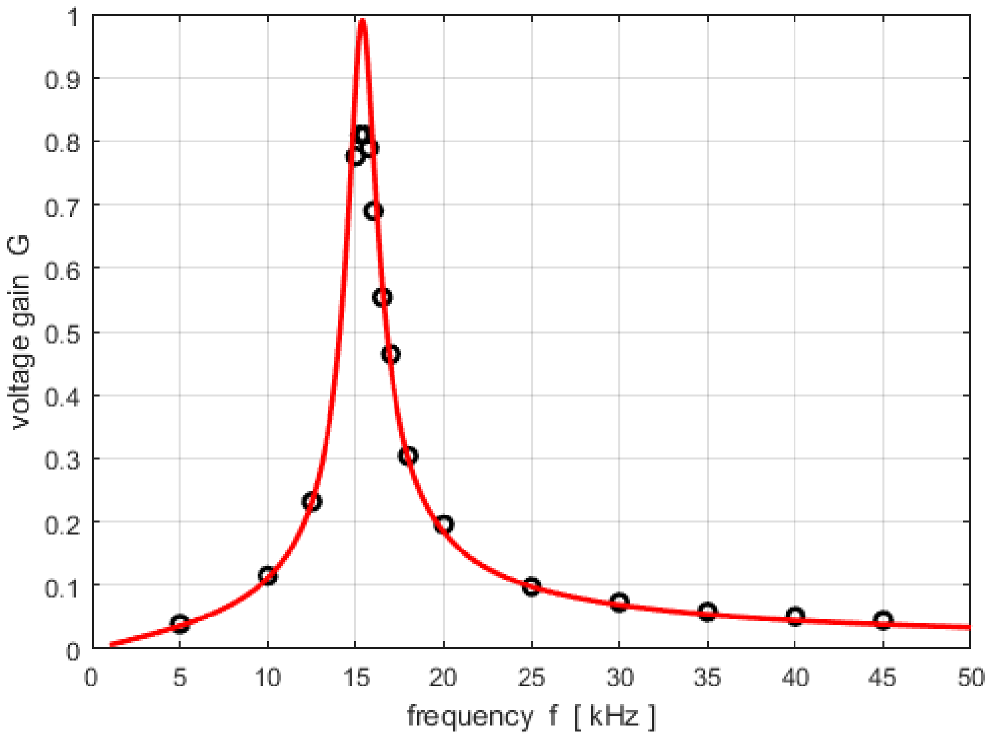

Fig. 18. The fit of the model to the measurements with $R_L = 0\ \Omega$.

$f_0 = 15377\ \text{Hz} \quad f_{peak} = 15379\ \text{Hz} \quad \Delta f = 1539\ \text{Hz} \quad Q = 9.99$

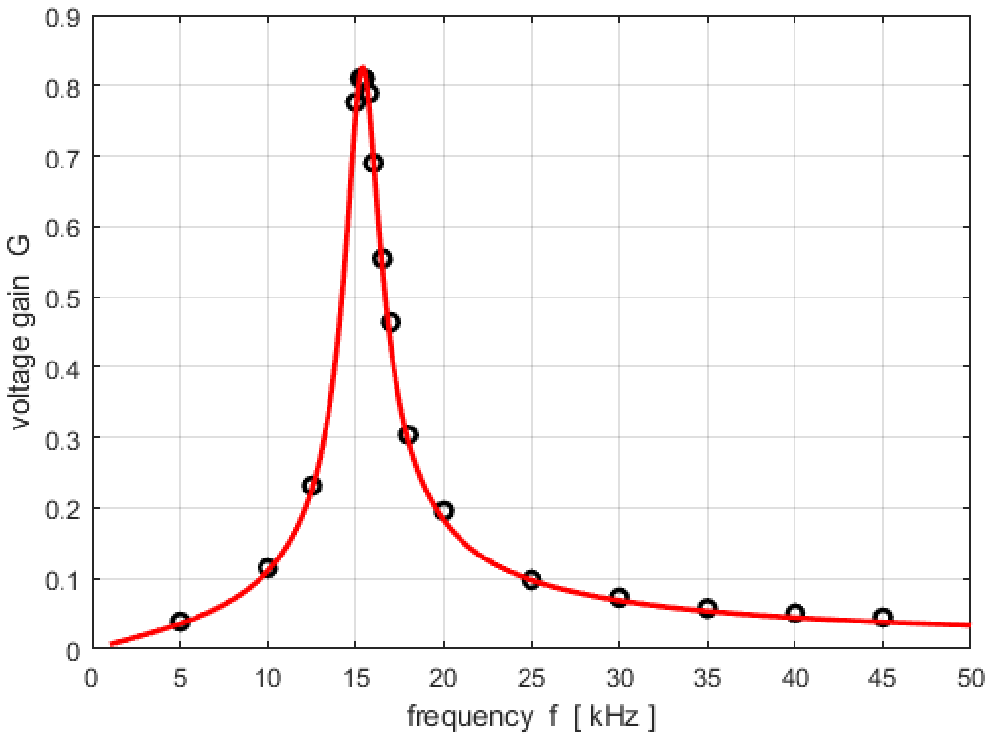

Fig. 19. The fit of the model to the measurements with $R_L = 20.0\ \Omega$.

$$f_0 = 15377\text{ Hz} \quad f_{peak} = 15389\text{ Hz} \quad \Delta f = 1843\text{ Hz} \quad Q = 8.34$$

The model with $R_L = 20.0\ \Omega$ gives an excellent fit to the measurements. The DC resistance measurement of the coil resistance measured with a multimeter was 12.5 Ω.

The resonance frequency is slightly higher than predicted from the relationship

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

The bandwidth is increased and the Q of the resonance circuit is lower.

If you consider the simplicity of the code in the Matlab script to model resonance circuits, this computational approach has many advantages compared with the traditional algebraic approach.