DOING PHYSICS WITH MATLAB

ELECTROMAGNETISM USING THE FDTD METHOD

[1D] Propagation of Electromagnetic Waves

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Matlab Download Directory

ft_03.m   ft_sources.m

Download and run the script **ft_03.m**. Carefully inspect the script to see how the FDTD method is implemented. Many variables can be changed throughout the script, for example, type of excitation signal, boundary conditions, time scales, properties of the medium.

The script **ft_03.m** is a very versatile program to investigate many aspects of the propagation of electromagnetic waves through dielectric media. You can investigate: free space propagation; propagation in different dielectric media; propagation in lossy dielectric media; reflection and refraction (transmission at an interface); interference effects; resonance.

The **Finite-Difference Time-Domain Method** (**FDTD**) is one of the most popular techniques used in solving problems in electromagnetism because it is very easy to write the computer code even for three-dimensional problems. The method was first proposed by K. Yee in the early 1970s. In this document, solutions to Maxwell's equation will be given for the one-dimensional propagation of electromagnetic waves generated from a point source.

MAXWELL'S EQUATIONS and the FDTD Method

The theory on which the FDTD is simple. To solve problems in electromagnetism, you simply discretise in both space and time the Maxwell's curl equations with a central difference approximation.

Maxwell's equations predict the existence of electromagnetic waves that propagate through free space at the speed of light c_0. The electric field and the magnetic field are time dependent and influence each other - a time varying magnetic field induces a time varying electric field and the time varying electric field induces a time varying magnetic field and the process just continues.

The time-dependent Maxwell's curl equations in a non-magnetic lossy dielectric material with a dielectric constant ε_r and the losses determined by the medium's conductivity σ are

(1a) $$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_r\,\varepsilon_0}\left(\nabla \times \vec{H} - \vec{J}\right)$$

(1b) $$\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_0}\nabla \times \vec{E}$$

where the current density $\vec{J}$ is $\vec{J} = \sigma\,\vec{E}$.

For the one-dimensional case where a plane electromagnetic wave propagates in the Z direction due to a time varying electric field component E_x and a magnetic field component H_y, Maxwell's curl equations reduce to

(2a) $$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_r\,\varepsilon_0}\frac{\partial H_y}{\partial z} - \frac{\sigma}{\varepsilon_r\,\varepsilon_0}E_x$$

(2b) $$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_x}{\partial z}$$

This mode of propagation is called a **TEM wave** (electric field polarized in X direction with $E_z = 0$ and $H_z = 0$).

The values of ε_0 and μ_0 differ by several orders of magnitude and hence E_x and H_y will also differ by several orders of magnitude when E_x and H_y are measured in S.I. units. This problem can be overcome by making a change of variable where E is replaced by a scaled value E_S for the electric field

(3) $$E = \sqrt{\frac{\mu_0}{\varepsilon_0}} E_s$$

which gives

(4a) $$\frac{\partial E_{sx}}{\partial t} = -\frac{1}{\varepsilon_r \sqrt{\varepsilon_0 \mu_0}} \frac{\partial H_y}{\partial z} - \frac{\sigma}{\varepsilon_r \varepsilon_0} E_{sx}$$

(4b) $$\frac{\partial H_y}{\partial t} = -\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{\partial E_{sx}}{\partial z}$$

We can approximate both the spatial and temporal partial derivatives using the **central difference method**

$$\frac{E_{sx}\left(t+\Delta t/2,z\right)-E_{sx}\left(t-\Delta t/2,z\right)}{\Delta t}=$$

(5a)
$$-\frac{1}{\varepsilon_r\sqrt{\varepsilon_0\,\mu_0}}\frac{H_y\left(t,z+\Delta z/2\right)-H_y\left(t,z-\Delta z/2\right)}{\Delta z}$$
$$-\frac{\sigma}{2\,\varepsilon_r\,\varepsilon_0}\left(E_{sx}\left(t+\Delta t/2,z\right)+E_{sx}\left(t-\Delta t/2,z\right)\right)$$

To find the latest value for the electric field, equation 5a is rearranged to give

(5b)

$$E_{sx}\left(t+\Delta t/2,z\right)=\frac{\left(1-\dfrac{\Delta t\,\sigma}{2\,\varepsilon_r\,\varepsilon_0}\right)}{\left(1+\dfrac{\Delta t\,\sigma}{2\,\varepsilon_r\,\varepsilon_0}\right)}E_{sx}\left(t-\Delta t/2,z\right)$$
$$-\frac{\Delta t}{\varepsilon_r\sqrt{\varepsilon_0\,\mu_0}\left(1+\dfrac{\Delta t\,\sigma}{2\,\varepsilon_r\,\varepsilon_0}\right)\Delta z}\left(H_y\left(t,z+\Delta z/2\right)-H_y\left(t,z-\Delta z/2\right)\right)$$

and equation (4b) becomes

(5c)

$$H_y\left(t+\Delta t, z+\Delta z/2\right)=H_y\left(t, z+\Delta z/2\right)$$
$$-\left(\frac{\Delta t}{\sqrt{\varepsilon_0\,\mu_0}\,\Delta z}\right)\left(E_{sx}\left(t+\Delta t/2, z+\Delta z\right)-E_{sx}\left(t+\Delta t/2, z\right)\right)$$

We can simulate an electromagnetic wave propagating from one medium to another by making both the relative dielectric $\varepsilon_r(z)$ and conductivity $\sigma(z)$ functions of z. To simply the coding, we can define a series of functions

(6a) $$K_1(z)=1-\frac{\Delta t\,\sigma(z)}{2\,\varepsilon_r(z)\,\varepsilon_0}$$

(6b) $$K_2(z)=1+\frac{\Delta t\,\sigma(z)}{2\,\varepsilon_r(z)\,\varepsilon_0}$$

(6c) $$K_3(z)=\frac{K_1}{K_2}$$

(6d) $$K_4=\frac{\Delta t}{\sqrt{\varepsilon_0\,\mu_0}\,\Delta z}=\frac{\Delta t\,c_0}{\Delta z}$$

(6e) $$K_5(z)=\frac{K_4}{\varepsilon_r(z)\,K_2}$$

For stability of the iterative method is often given by the **Courant Condition**

(7) $$\Delta t \le \frac{\Delta z}{\sqrt{D}\, c_0} = \frac{\Delta z \sqrt{\varepsilon_0\, \mu_0}}{\sqrt{D}}$$

where *D* is the dimension of the simulation and we will take the equality sign for the stability condition. Thus, a given cell size or grid spacing Δz determines the time step Δt in a simulation or vice-versa

(8)
$$\Delta t = \frac{\Delta z}{\sqrt{D}\, c_0} \quad \rightarrow \quad K_4 = \frac{1}{\sqrt{D}}$$
$$\Delta z = \sqrt{D}\, c_0\, \Delta t$$

The default value used in the simulation is ***D* = 4** giving

(9) $$\Delta t = \frac{\Delta z}{2\, c_0} \qquad \Delta z = 2\, c_0\, \Delta t \qquad K_4 = \frac{1}{2}$$

Substituting equations 6 into equations 5b and 5c gives

(10a)

$$E_{sx}\left(t+\Delta t/2,z\right)=K_3\ E_{sx}\left(t-\Delta t/2,z\right)$$
$$-K_5\left(H_y\left(t,z+\Delta z/2\right)-H_y\left(t,z-\Delta z/2\right)\right)$$

(10b)

$$H_y\left(t+\Delta t,z+\Delta z/2\right)=H_y\left(t,z+\Delta z/2\right)$$
$$-K_4\left(E_{sx}\left(t+\Delta t/2,z+\Delta z\right)-E_{sx}\left(t+\Delta t/2\right),z\right)$$

Equations 10a and 10b are interleaved, the new value of E_{sx} at position z is calculated from the previous value of E_{sx} at position z and the most recent pair values of H_y at $\left(z-\Delta z/2\right)$ and $\left(z+\Delta z/2\right)$. H_y is calculated at $\left(z+\Delta z/2\right)$ from its previous value at $\left(z+\Delta z/2\right)$ and the most recent values of E_{sx} at z and $\left(z+\Delta z\right)$.

This **interleaving** is at the heart of the FDTD method, that is, the equations are solved in a **leap-frog** manner where the electric field is solved at a given instant in time, then the magnetic field is solved at the next instant in time, and the process is repeated over and over again.

To write the mscript to solve iteratively equations 10a and 10b, we need to assign indices for time ct and position cz, where

$$ct = 1, 2, 3, \ldots, nt$$

$$cz = 1, 2, 3, \ldots, nz$$

For the electric field E_{sx}

Time: $t - \Delta t / 2 \rightarrow ct - 1 \quad t + \Delta t / 2 \rightarrow ct$

Position: $z - \Delta z \rightarrow cz - 1 \quad z \rightarrow cz \quad z + \Delta z \rightarrow cz + 1$

For the magnetic field H_y

Time: $t \rightarrow ct \quad t + \Delta t \rightarrow ct + 1$

Position:

$$z - \Delta z / 2 \rightarrow cz - 1 \quad z + \Delta z / 2 \rightarrow cz \quad z + 3\Delta z / 2 \rightarrow cz + 1$$

Equations 10a and 10b can now be expressed in a format that is now straight forward to write the computer code

(11a)

$$E_{sx}\left(ct,cz\right)=K_3(cz)\,E_{sx}\left(ct-1,cz\right)$$
$$-K_5(cz)\,H_y\left(ct-1,cz\right)+K_5(cz-1)\,H_y\left(ct-1,cz-1\right)$$

(11b)

$$H_y\left(ct,cz\right)=H_y\left(ct-1,cz\right)-K_4\left(E_{sx}\left(ct,cz+1\right)-E_{sx}\left(ct,cz\right)\right)$$

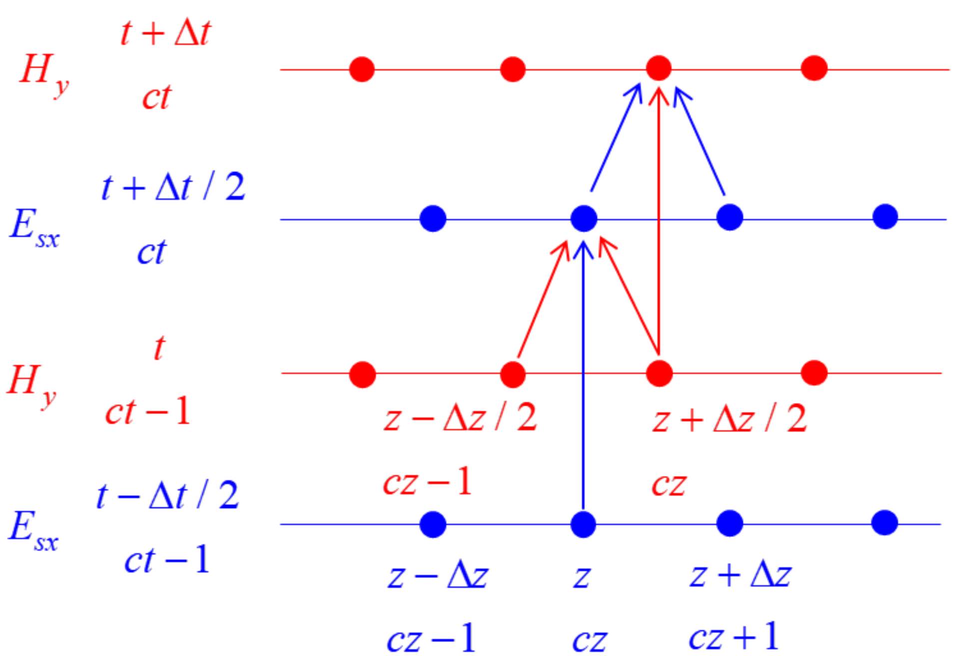

Fig. 1. Interleaving of the E_{sx} and H_y fields in space and time in the FDTD method.

BOUNDARY CONDITIONS

A fundamental assumption in the FDTD method is that in calculation the E and H fields, we need to know the surrounding H and E field values, but at the edges of the Z space we do not have values for E and H. So, it is important that the boundary values for E and H are specified at each time step.

The boundary condition where E = 0 is called a **perfect electric conductor boundary** (PEC). A **perfect magnetic conductor boundary** (PMC) is when H = 0 is set as the boundary condition. This means that when a pulse arrives at the ends of the Z space, the boundary conditions that are imposed on the solution results in the reflection of both the electric and magnetic fields.

Absorbing boundary conditions

We can stop the reflections at the boundary by applying absorbing boundary conditions. We can solve this problem by assuming that there are no sources outside the Z space and that the wave propagates outward across the boundary. From the stability condition given by equation 9 with $D = 4$

(9) $$\Delta t = \frac{\Delta z}{2\,c_0} \qquad \Delta z = 2\,c_0\,\Delta t$$

it takes two time steps for the wave to propagate that from one grid position to the next in free space. Hence, to apply absorbing boundary conditions at the ends of the Z space, the values of the fields at the boundaries of the Z space are set to the values of the adjacent z position two time steps earlier. In terms of the space cz and time ct indices, the absorbing boundary conditions are:

(13) $$\begin{aligned} E_x\left(1, ct\right) &= E_x(2, ct-2) \\ E_x\left(nz, ct\right) &= E_x(nz-1, ct-2) \\ H_y\left(1, ct\right) &= H_y(2, ct-2) \\ H_y\left(nz, ct\right) &= E_x(nz-1, ct-2) \end{aligned}$$

These boundary conditions are easy to code. We need to simply store the values for the fields adjacent to the end points of the Z space for the previous two time steps.

Excitation of the propagating *E* and *H* fields

A variety of functions at any grid point can be used as the source of the propagating electromagnetic wave. The mscript **fd_sources.m** was used to create the plots of the source functions shown in figures 2, 3 and 4.

Gaussian pulse

A Gaussian pulse in the electric field at a grid point produces an electromagnetic wave pulse that propagates away in both directions from a fixed source point.

The values of E_{sx} and H_y are calculated by separate loops due to the interleaving of the E_{sx} and H_y values. After E_{sx} has been calculated, the E_{sx} value at the source point is over-written by the value calculated from the Gaussian source function when its value is greater than some threshold value. This is referred to as a **hard source** because a specific value is imposed on E_{sx} on the FDTD grid. In wave impinging upon the hard source will lead to reflections.

The Gaussian pulse is given by equation 14

$$(14) \qquad E_{sx}(z_S) = A\exp\left(-\frac{0.5(t_0 - ct)}{s}\right)$$

where z_S is the index specifying the location of the point source, A is the maximum height of the pulse, t_0 determines the time step index for the peak value of the pulse, s is the spread of the pulse and ct is the index for the time step.

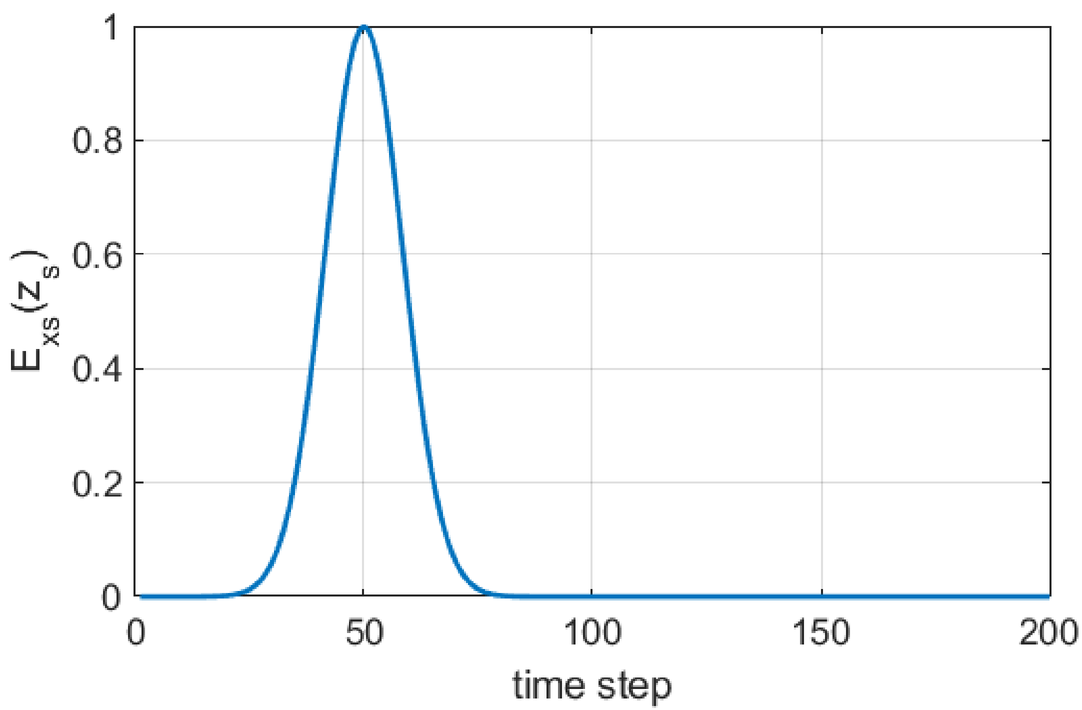

Fig. 2. Hard source: Gaussian time variation in E_{sx} at source point. (A = 1, s = 12, t_0 = 40).

Sinusoidal Excitation

A continuous sinusoidal source (equation 16) can act to excite the propagation of the electromagnetic wave along the Z axis.

(16) $$E_{sx}(z_S) = A\sin\left(2\pi\, f_s\, ct\,\Delta t\right)$$

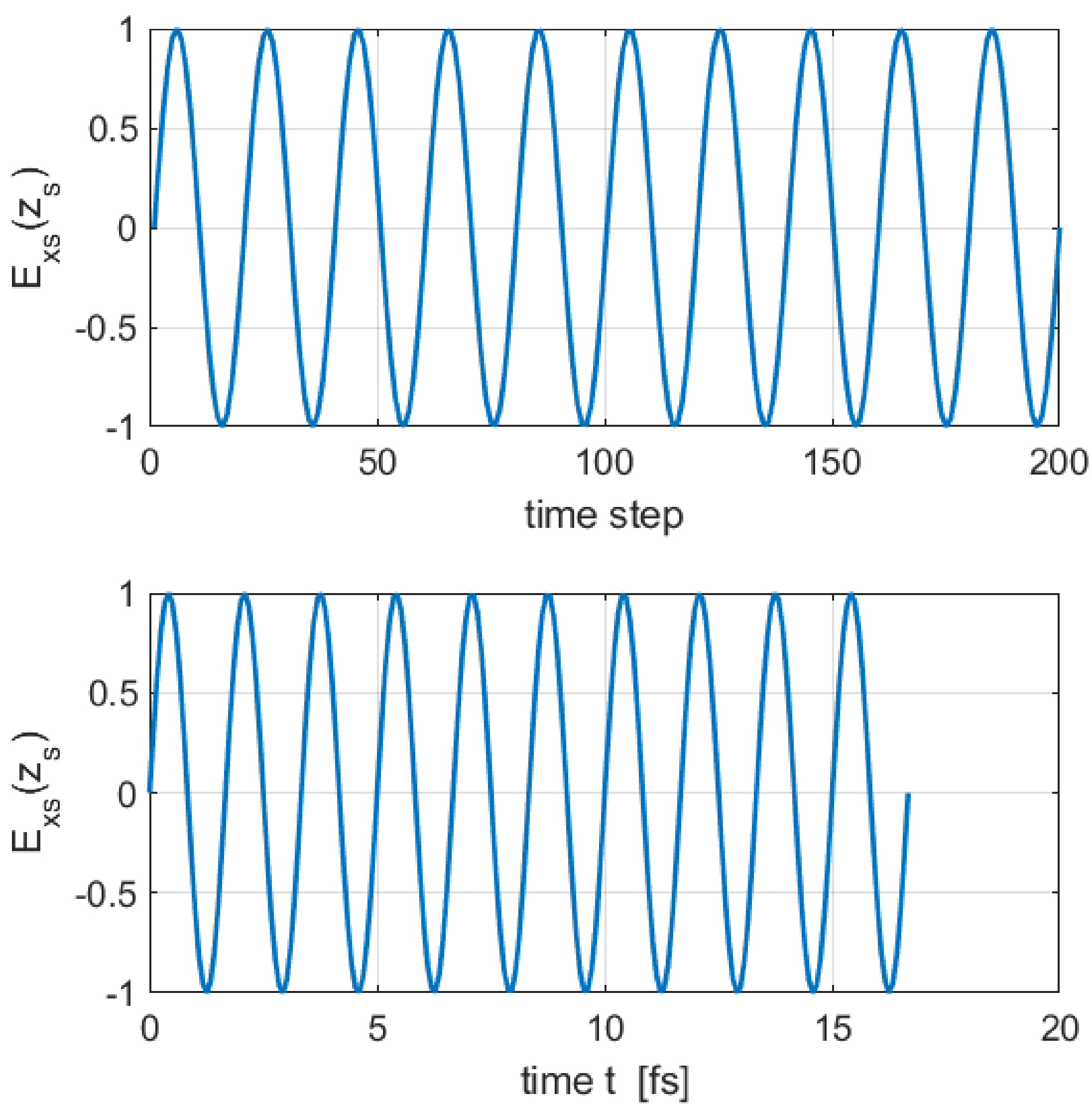

Fig. 3. Sinusoidal excitation signal.

Modulated Gaussian pulse

A modulated Gaussian pulse given by equation 15 act as a source at the point z_S.

$$(15)\quad E_{sx}(z_S) = A\sin\left(2\pi\ f_s\,ct\,\Delta t\right)\exp\left(-\frac{0.5\left(t_0 - ct\right)}{s}\right)$$

The frequency of the modulation signal is f_s.

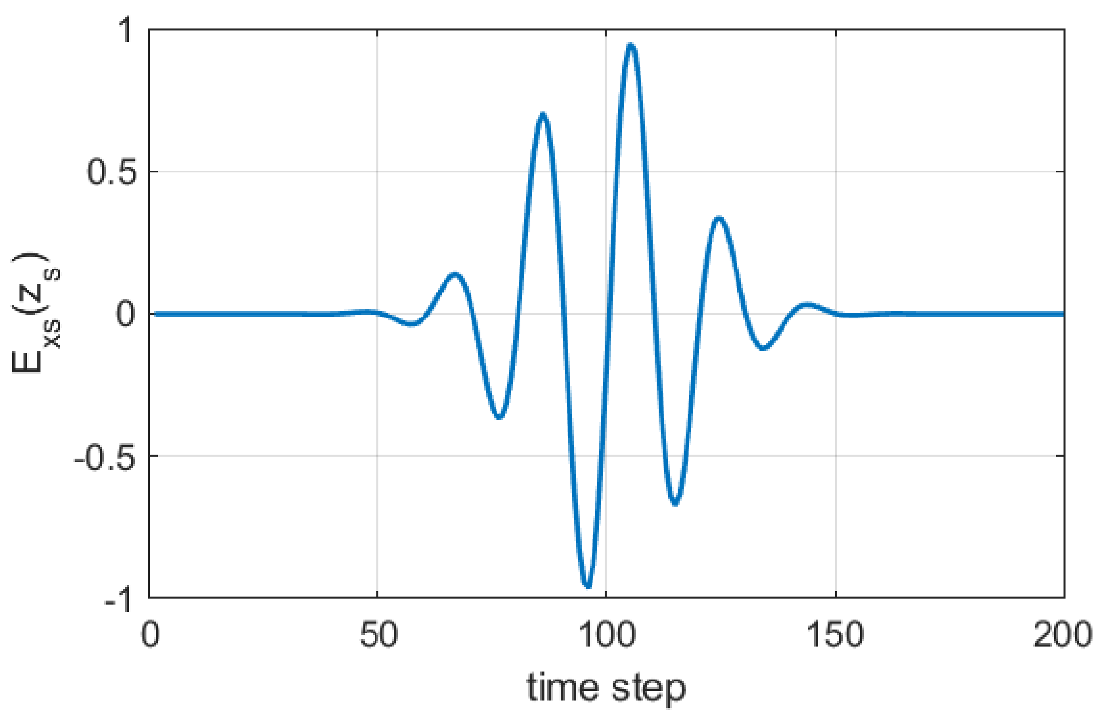

Fig. 4. Modulated Gaussian pulse.

fd_sources.m

```
% INPUTS ==================================================
% Number of time steps
    nt = 200;
% Amplitude of source signal
    A = 1;
% Width of source signal
    s = 12;
% Signal frequency
    fS = 3e8/500e9;
% Time for peak signal
    ct0 = 100;
% wavelength
    wL0 = 500e-9;

% CALCULATIONS ============================================
% speed of light / frequency / period
   c0 = 3e8;
   f0 = c0/wL0;
   T0 = 1/f0;
   tMax = 10*T0;
   t = linspace(0,tMax,nt);
   dt = t(2)-t(1);
      ct = 1:nt;


% GAUSSIAN PULSE
  EG = A.*exp(-(0.5.*(ct0 - ct)./s).^2);
% SINE FUNCTION
  ES = sin(2*pi*f0*t);
% MODULATED GAUSSIAN SIGNAL
  EMG = EG .* ES;
```

MATLAB script ft_03.m

The finite difference time domain (FDTD) uses a centre-difference representation of the continuous partial differential equations to create iterative numerical models of electromagnetic wave propagation by solving Maxwell's equations in the time domain. Maxwell's equations are discretized in time and space and a leap-frog algorithm is used to find the E_x-field and H_y-field as functions of time and space.

The number of time steps Nt is varied to change the simulation time.

The number of spatial grid points is specified by the variable Nz. Generally Nz is fixed at the default value Nz = 400.

Time and position are **not** independent quantities as shown by equation 7.

$$(7) \qquad \Delta t \le \frac{\Delta z}{\sqrt{D}\, c_0} = \frac{\Delta z \sqrt{\varepsilon_0\, \mu_0}}{\sqrt{D}}$$

To specify the Z axis, the wavelength associated with a sinusoidal wave is given by the variable lambda.

In the simulations using the script, a point source is used to excite an electromagnetic wave that propagates along the Z axis. You can select the source excitation by setting the value of the variable flagS.

flagS = 1 Gaussian pulse excitation

flagS = 2 Sinusoidal excitation

flagS = 3 Modulated sinusoidal excitation with a Gaussian envelope

The source is specified by the inputs: zS (Z index for location of excitation point); A (amplitude); width (width of the Gaussian pulse in time steps); centre (centre of pulse in time steps).

The properties of the media are specified by the relative permittivity (dielectric constant) eR and the conductivity S. The default for the program is to have a uniform medium or two uniform media where the boundary occurs at the grid position given by M2. You can specify the electrical properties of the Z space by specifying the relativity permittivity and conductivity for a range of grid points. This is not done in the INPUT section of the script.

```
% ELECTRICAL PROPERTIES OF MEDIA
   eR = ones(1,Nz).* eR1;  % Relative permittivity
   indexR = round(200:200+12.5/8);
   eR(indexR) = eR2;
```

We can monitor the time evolution of the E-field and H-field at five Probe positions along the Z axis which is specified by the variable cP. You can easily change the positions of the Probes.

The variables limE and limH are used to change the Y limits for the plots of the E-field and H-field as functions of time in figure 1.

The boundary conditions are specified by the variable flagBC.

```
% BOUNDARY CONITIONS --------------------------------------
---------------
%     flagBC = 1    absorbing
%     flagBC = 2    Perfect electric conductor PEC at end
                    boundary only
%     flagBC = 3    Perfect magnetic conductor PMC at end
                    boundary only
%     flagBC = 4    Perfect electric conductor PEC at both
                    boundaries
   flagBC = 1;
```

An animation of figure can be saved as a gif file using the variable f_gif = 1.

The results of the modelling are shown in Figure Windows.

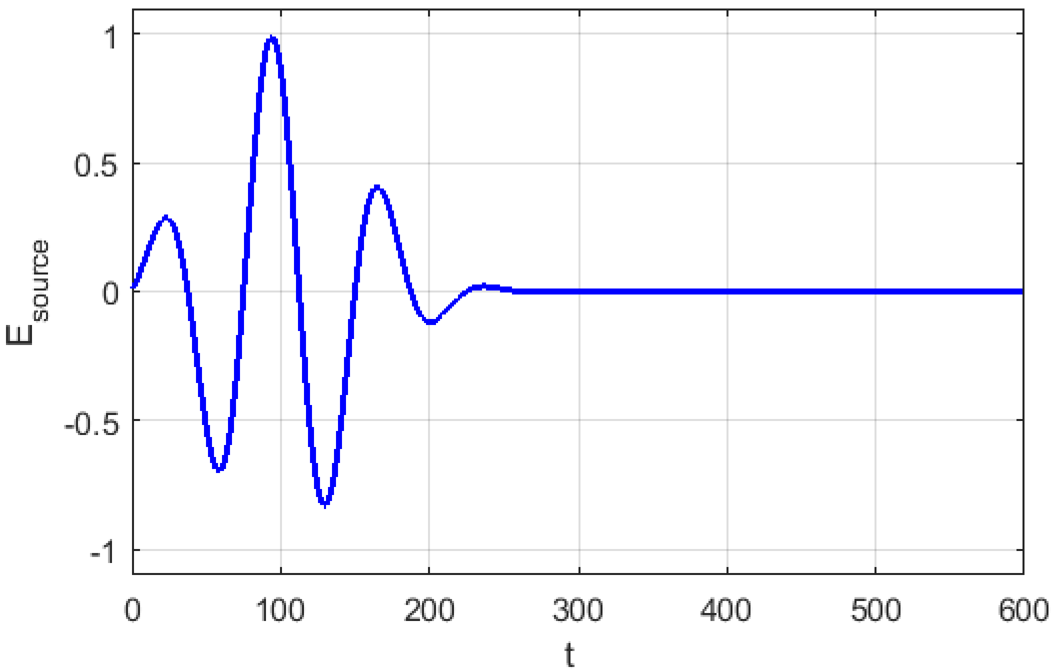

Fig. 5. Plot of the source function (Modulate Gaussian Pulse: width = 50 and centre = 100)

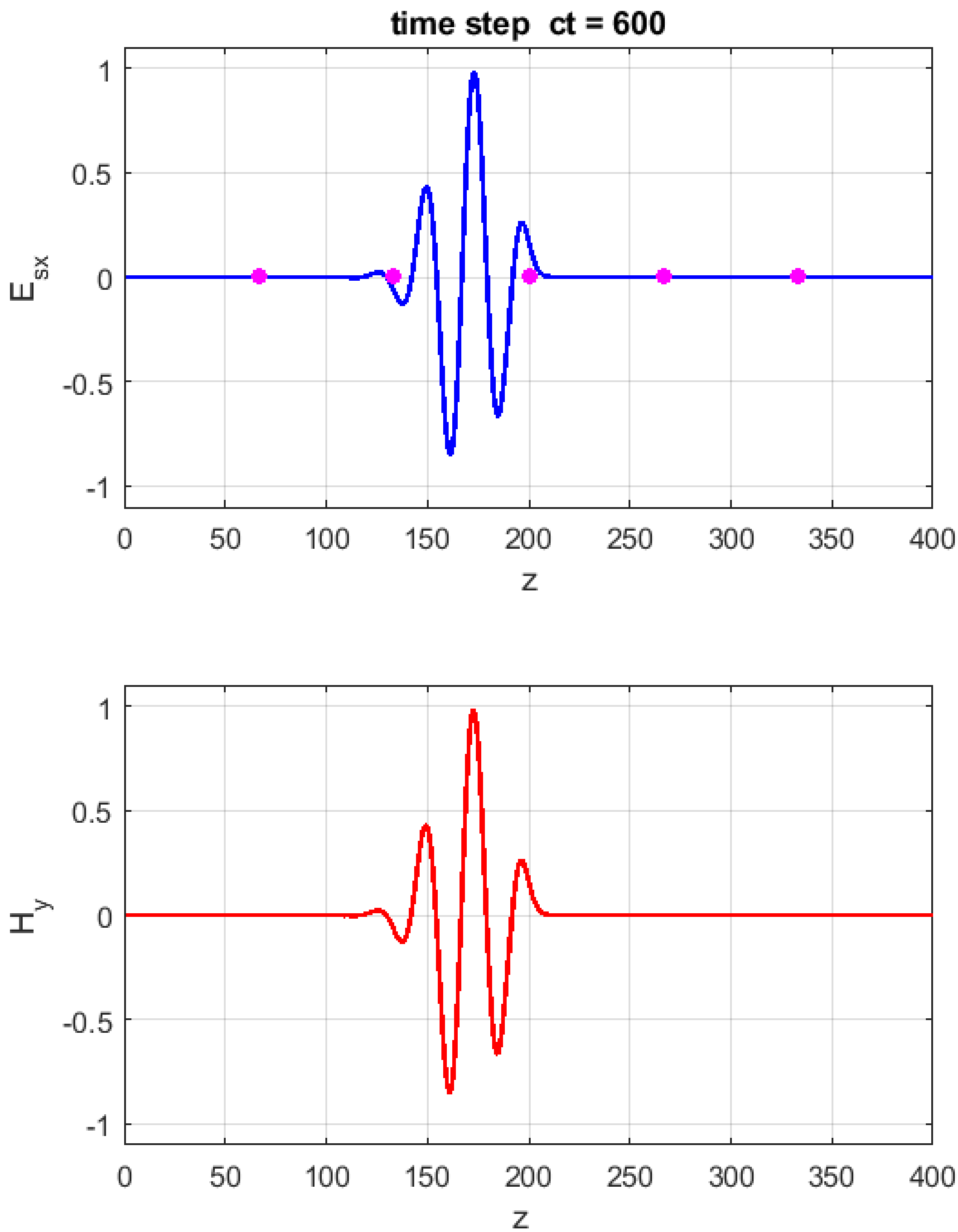

Fig. 6. The E-field and H-field after 600 time steps. The magenta dots show the positions of the 5 Probes.

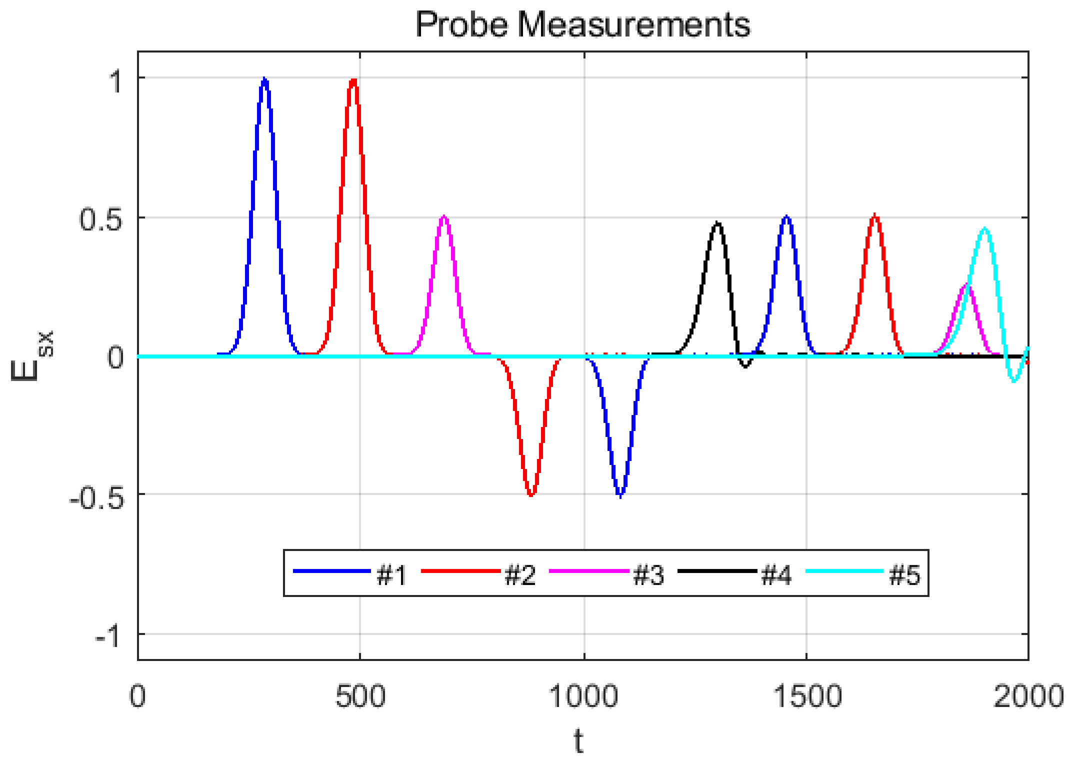

Fig. 7. EM pulse hitting a medium with a different dielectric constant. Pulse: width = 25 and centre = 100. Medium 1 is free space and Medium 2 has a dielectric constant of 9.

time step dt = 1.90e-11 s   cell size dz = 1.71e-02 m

PROBE positions [grid points]

z_1 = 67 z_2 = 133 z_3 = 200 z_4 = 267 z_5 = 333

Velocity between Probes #1 and #2 v_{12} = 2.98e+08 m/s

Velocity between Probes #4 and #5 v_{45} = 9.92e+07 m/s

Amplitudes

A_1 = 1.000 A_2 = 1.000 A_3 = 0.502 A_4 = 0.480 A_5 = 0.456

Fig. 8. Summary of the results for EM pulse shown in figure 7 the hitting the boundary. The pulse travels at a speed of $c_0/3$ in Medium 2 which has a dielectric constant of 9.

Reference

Dennis M Sullivan, *Electromagnetic Simulation Using the FDTD Method*. IEEE Press.