DOING PHYSICS WITH MATLAB

VECTOR ANANYSIS

Ian Cooper

School of Physics, University of Sydney

ian.cooper@sydney.edu.au

DOWNLOAD DIRECTORY FOR MATLAB SCRIPTS

cemVectorsA.m

Inputs: Cartesian components of the vector V

Outputs: cylindrical and spherical components and [3D] plot of vector

cemVectorsB.m

Inputs: Cartesian components of the vectors A B C

Outputs: dot products, cross products and triple products

cemVectorsC.m

Rotation of XY axes around Z axis to give new of reference X'Y'Z'.

Inputs: rotation angle and vector (Cartesian components) in XYZ frame

Outputs: Cartesian components of vector in X'Y'Z' frame

mscript can be modified to calculate the rotation matrix for a [3D] rotation and give the Cartesian components of the vector in the X'Y'Z' frame of reference.

SPECIFYING a [3D] VECTOR

A **scalar** is completely characterised by its magnitude, and has no associated direction (mass, time, direction, work). A scalar is given by a simple number.

A **vector** has both a magnitude and direction (force, electric field, magnetic field). A vector can be specified in terms of its Cartesian or cylindrical (polar in [2D]) or spherical coordinates.

Cartesian coordinate system (XYZ right-handed rectangular: if we curl our fingers on the right hand so they rotate from the X axis to the Y axis then the Z axis is in the direction of the thumb).

A vector $\vec{V}$ in specified in terms of its X, Y and Z Cartesian components

$$\vec{V}\left(V_x, V_y, V_z\right) \qquad \vec{V} = V_x\,\hat{i} + V_y\,\hat{j} + V_z\,\hat{k}$$

where $\left(\hat{i}, \hat{j}, \hat{k}\right)$ are unit vectors parallel to the X, Y and Z axes respectively.

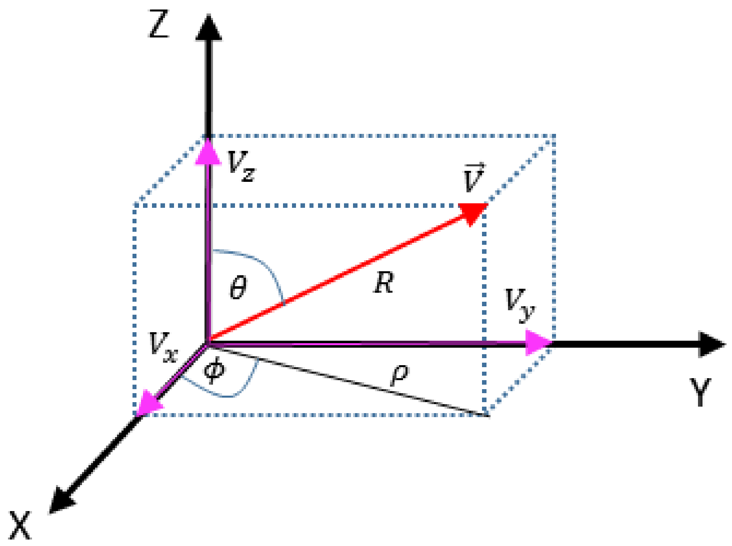


Fig. 1. Specifying a vector in an orthogonal coordinate system.

The **polar angle** θ is the angle down from the Z axis to the vector $\vec{V}$.

The **azimuthal angle** ϕ is the angle around from the X axis.

Polar angle $\quad 0 \le \theta \le \pi$

Azimuthal angle $\quad 0 \le \phi \le 2\pi \quad$ or $\quad -\pi \le \phi \le +\pi$

Angles can be measured in radians or in degrees where $2\pi \text{ rad} = 360^{\circ}$

You can use the Matlab functions **rad2deg** and **deg2rad** for the conversions between radians and degrees

`deg2rad(30)` → 30° = 0.5236 rad

`rad2deg(pi)` → π = 180°

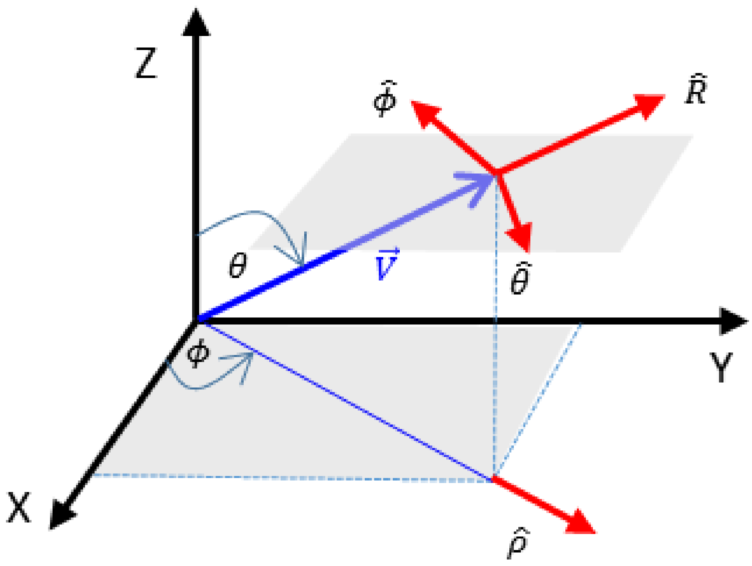


Fig. 2. The unit vectors $\hat{R}$, $\hat{\theta}$, $\hat{\phi}$, $\hat{\rho}$ pointing in the direction of an increase in the corresponding coordinate.

Cartesian components $\vec{V}\left(V_x, V_y, V_z\right)$ $\vec{V} = V_x\,\hat{i} + V_y\,\hat{j} + V_z\,\hat{k}$

Cylindrical components $\vec{V}\left(V_\rho, V_\phi, V_z\right)$ $\vec{V} = V_\rho\,\hat{\rho} + V_\phi\,\hat{\phi} + V_z\,\hat{k}$

Spherical components $\vec{V}\left(V_R, V_\theta, V_\phi\right)$ $\vec{V} = V_R\,\hat{R} + V_\theta\,\hat{\theta} + V_\phi\,\hat{\phi}$

Magnitudes

$$\left|\vec{V}\right| \equiv V \equiv R = \sqrt{V_x^{\,2} + V_y^{\,2} + V_z^{\,2}}$$

$$\rho = \sqrt{V_x^{\,2} + V_y^{\,2}}$$

Relationship between coordinates from figure 2

$$V_x = R\sin\theta\cos\phi \qquad V_y = R\sin\theta\sin\phi \qquad V_z = R\cos\theta$$

$$V_x = \rho\cos\phi \qquad V_y = \rho\sin\phi \qquad V_z = V_z$$

$$\tan\phi = \frac{V_y}{V_x} \qquad \tan\theta = \frac{\rho}{V_z} \qquad \cos\theta = \frac{V_z}{R}$$

Spherical coordinates

$$\hat{R} = \sin\theta\ \cos\phi\ \hat{i} + \sin\theta\ \sin\phi\ \hat{j} + \cos\theta\ \hat{k}$$
$$\hat{\theta} = \cos\theta\ \cos\phi\ \hat{i} + \cos\theta\ \sin\phi\ \hat{j} - \sin\theta\ \hat{k}$$
$$\hat{\phi} = -\sin\phi\ \hat{i} + \cos\theta\ \sin\phi\ \hat{j}$$

Cylindrical coordinates

$$\hat{\rho} = \cos\phi\ \hat{i} + \sin\phi\ \hat{j}$$
$$\hat{\phi} = -\sin\phi\ \hat{i} + \cos\theta\ \sin\phi\ \hat{j}$$
$$\hat{z} = \hat{z}$$

Matlab

Vector $\vec{V} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

row vector `V = [3 5 -6]`

column vector `V = [3; 5; -6]`

magnitudes `R = norm(V)` `rho = sqrt(V(1)^2+V(2)^2)`

azimuthal angle $0 \le \phi \le 2\pi$

```
phi = atan2(V(2),V(1));
if phi < 0, phi = phi + 2*pi; end
```

polar angle

```
theta = acos(V(3)/R);
thetaD = acosd(V(3)/R);
```

Matlab: changing orthogonal systems

You can also use MATLAB functions to make the conversion between Cartesian, polar, cylindrical, and spherical coordinate systems easy.

Cartesian components $V_x \quad V_y \quad V_z$ `Vx Vy Vz`

Cylindrical components $V_\rho \quad V_\phi \quad V_z$ `Vrho Vphi Vz`

Spherical components $V_R \quad V_\phi \quad V_\theta$ `VR Vphi Vtheta`

where the angles `Vphi` and `Vtheta` are in radians

[2D]

```
[Vphi Vrho] = cart2pol(Vx,Vy)
[Vx Vy] = pol2cart(Vphi, Vrho)
```

[3D]

```
[Vphi Vrho Vz] = cart2pol(Vx, Vy, Vz)
[Vx Vy Vz] = pol2cart(Vphi, Vrho, Vz)

[Vtheta Vphi VR] = cart2sph(Vx, Vy, Vz)
[Vx Vy Vz] = sph2cart(Vtheta, Vphi, VR)

[Vphi Vrho Vz] = sph2pol(Vtheta, Vphi, VR)
[Vtheta Vphi VR] = pol2sph(Vphi, Vrho, Vz)
```

Sample results

V_x	V_y	V_z	V_ρ	V_R	V_ϕ	V_θ
3	5	-6	5.831	8.367	1.03 rad 59.04°	0.80 rad -45.82°
$\sqrt{3/2}$ 1.2247	$1/\sqrt{2}$ 0.7071	$\sqrt{2}$ 1.4142	$\sqrt{2}$ 1.4142	2.0000	0.52 rad 30.00°	0.79 rad 45.00°
$\sqrt{3/2}$ 1.2247	$-1/\sqrt{2}$ -0.7071	$-\sqrt{2}$ -1.4142	$\sqrt{2}$ 1.4142	2.0000	-0.52 rad -30.00°	0.79 rad -45.00°

Figure (1) gives a [3D] plot of a vector plus a summary of the input values for the Cartesian components and the calculated spherical and cylindrical components of the vector using the **mscript cemVectorsA.m**.

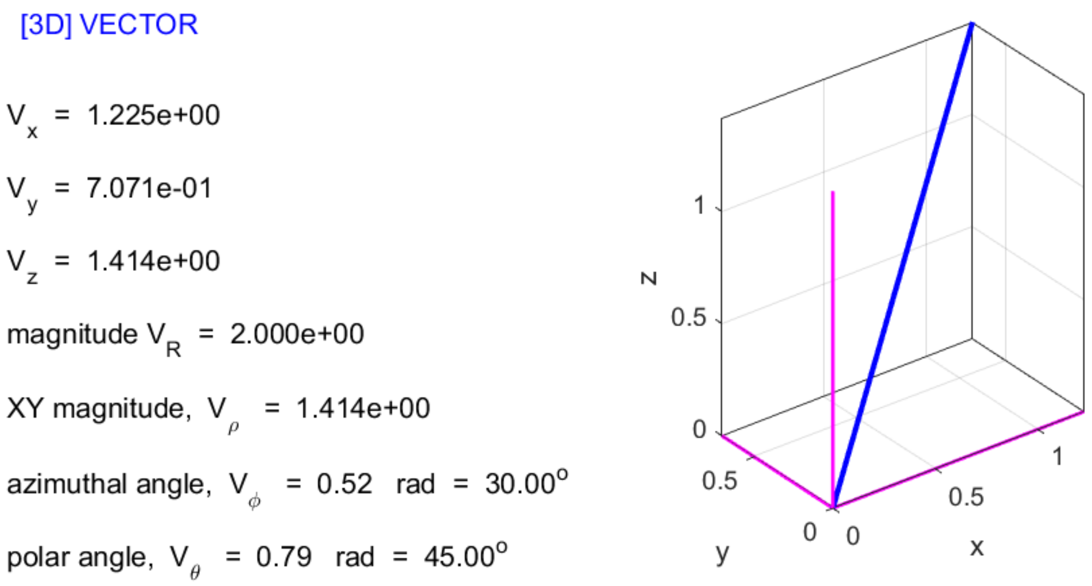


Fig. 1. Figure Window for a vector with inputs as the Cartesian components. **cemVectorsA.m**

Addition / Subtraction / Scalar multiplication

To add or subtract vectors, you add or subtract the components. For multiplication of a vector by a scalar, simply multiply each component by the number for the scalar.

For example: consider the vectors in Cartesian coordinates

$$\hat{A}(1,2,3) \quad \hat{B}(-1,-3,-5) \quad \hat{C}(2,4,-3)$$

$$\vec{V} = 3\vec{A} + \vec{B} - 2\vec{C}$$

$$\vec{V} = (3A_x + B_x - 2C_x)\hat{i} + (3A_y + B_y - 2C_y)\hat{j} - (3A_z + B_z - 2C_z)\hat{k}$$

$$\vec{V} = (3-1-4)\hat{i} + (6-3-8)\hat{j} - (9-5+6)\hat{k}$$

$$\vec{V} = -2\hat{i} - 5\hat{j} + 10\hat{k}$$

$$\vec{V}(-2,-5,10) \quad V_x = -2 \quad V_y = -5 \quad V_z = 10$$

Matlab Command Window

```
A = [1 2 3]

B = [-1 -3 -5]

C = [2 4 -3]

V = 3*A+B-2*C    →    V = [ -2   -5   10]
```

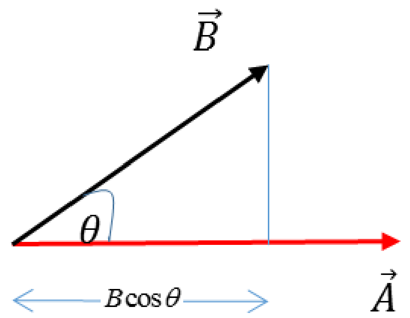
Dot product (scalar product) of two vectors

$$\vec{A} \bullet \vec{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

where θ is the angle between the two vectors when they are placed tail to tail

$\vec{A} \bullet \vec{B}$	scalar
$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$	commutative
$\vec{A} \bullet \left(\vec{B} + \vec{C}\right) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$	distributive

Geometrically $\vec{A} \bullet \vec{B}$ is the product of $\vec{A}$ times the projection of $\vec{B}$ along $\vec{A}$ or the product of $\vec{B}$ times the projection of $\vec{A}$ along $\vec{B}$



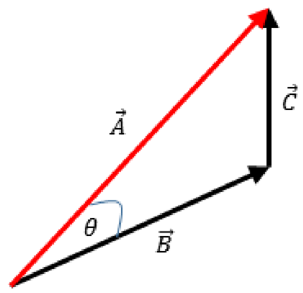
Angle between the two vectors $\cos\theta = \dfrac{\vec{A} \bullet \vec{B}}{AB} = \dfrac{A_x B_x + A_y B_y + A_z B_z}{AB}$

Law of cosines

$$\vec{A} = \vec{B} + \vec{C} \qquad \vec{C} = \vec{A} - \vec{B}$$

$$\vec{C} \bullet \vec{C} = \left(\vec{A} - \vec{B}\right) \bullet \left(\vec{A} - \vec{B}\right) = \vec{A} \bullet \vec{A} + \vec{B} \bullet \vec{B} - 2\vec{A} \bullet \vec{B}$$

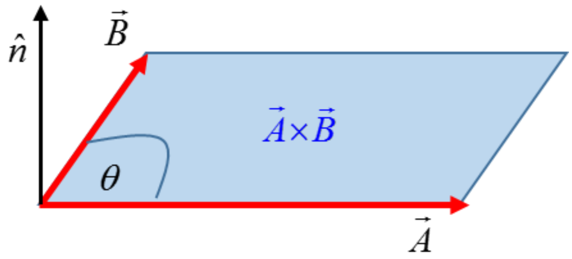
$$C^2 = A^2 + B^2 - 2AB\cos\theta$$



Matlab function **dot(A,B)** where A and B are row vectors of the same length.

Cross product or vector product of two vectors

$$\vec{A}\times\vec{B} = A\,B\sin\theta\;\hat{n}$$



where θ is the angle between the two vectors placed tail to tail and $\hat{n}$ is a unit vector that is normal to the plane defined by the two vectors and whose direction is determined by the right-hand rule (fingers curl from $\vec{A}$ to $\vec{B}$ then extended thumb points in direction of $\hat{n}$).

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A} \qquad \text{non-commutative}$$

$$\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B} + \vec{A}\times\hat{C} \qquad \text{distributive}$$

$$\vec{A}\,||\,\vec{B} \quad \Rightarrow \quad \theta = 0 \quad \vec{A}\times\vec{B} = 0$$

$$\vec{A}\times\vec{A} = 0$$

$$\vec{A}\perp\vec{B} \quad \Rightarrow \quad \theta = \pi/2\;rad \quad \left|\vec{A}\times\vec{B}\right| = A\,B$$

The cross product is the vector area of the parallelogram having $\vec{A}$ and $\vec{B}$ on adjacent sides.

Determinant form

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \left(A_y\,B_z - A_zB_y\right)\hat{i} + \left(A_z\,B_x - A_xB_z\right)\hat{j} + \left(A_x\,B_y - A_yB_x\right)\hat{k}$$

Matlab function **cross(A,B)** returns the cross product of the vectors A and B. A and B must be 3 element vectors.

Triple Products

Examples of triple products of three vectors

$$\vec{A}\bullet\left(\vec{B}\times\vec{C}\right)=\vec{B}\bullet\left(\vec{C}\times\vec{A}\right)=\vec{C}\bullet\left(\vec{A}\times\vec{B}\right)$$

$$\vec{A}\bullet\left(\vec{C}\times\vec{B}\right)=\vec{B}\bullet\left(\vec{A}\times\vec{C}\right)=\vec{C}\bullet\left(\vec{B}\times\vec{A}\right)$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\bullet\left(\vec{A}\bullet\vec{C}\right)-\vec{C}\bullet\left(\vec{A}\bullet\vec{B}\right)$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)\neq\left(\vec{A}\times\vec{B}\right)\times\vec{C}$$

Calculations of triple products are given in the following examples. You should compare the numerical outputs with the above relationships.

Results of running the mscript cemVectorsB.m

Inputs vectors

A = [1 0 1] B = [0 1 1] C = [2 4 5]

Outputs

Magnitudes of vectors `Amag = norm(A)`

Amag = 1.4142 Bmag = 1.4142 Cmag = 6.7082

Dot products `AdotB = dot(A,B)`

AdotB = 1 BdotA = 1 BdotC = 9 CdotA = 7

Cross products `AB = cross(A,B)`

AB = [-1 -1 1] BA = [1 1 -1] BC = [1 2 -2] CA = [4 3 -4]

Cross products: magnitudes `ABmag = norm(AB)`

ABmag = 1.7321 BAmag = 1.7321 BCmag = 3 CAmag = 6.4031

Angles between vectors `ABangle = asin(norm(AB) /(Amag * Bmag))`
`ABangle_deg = rad2deg(ABangle)`

ABangle = 1.0472 rad = 60.0000 deg

BAangle = 1.0472 rad = 60.0000 deg

BCangle = 0.3218 rad = 18.4349 deg

CAangle = 0.7409 rad = 42.4502 deg

Triple products (cross product of vectors A and B written as AB)

AdotBC = -1 `AdotBC = dot(A,cross(B,C))`

BdotCA = -1

CdotAB = -1

AdotCB = 1

BdotAC = 1

CdotBA = 1

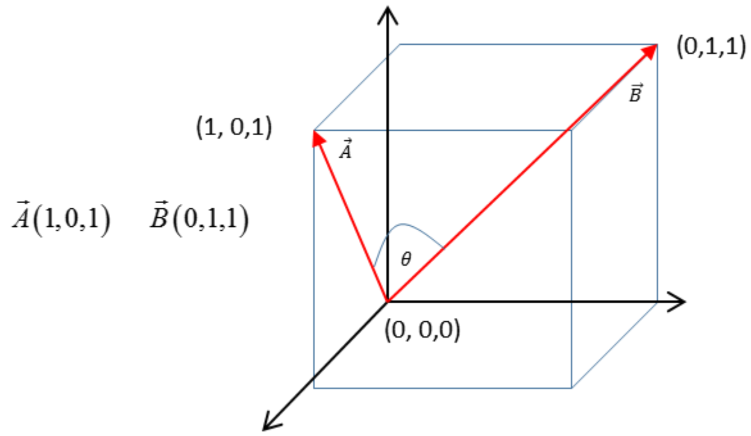
AcrossBC1 = [-2 3 2] `cross(A,cross(B,C))`

AcrossBC2 = [-2 3 2] `B .* dot(A,C) - C .* dot(A,B)`

AcrossBC = [-2 3 2] `AcrossBC = cross(A,cross(B,C))`

ABcrossC = [-9 7 -2] `ABcrossC = cross(cross(A,B),C)`

Example Find the angle between the face diagonals of a cube



⇒

The angle between the two vectors can be found from the cross product of the two vectors

$$\vec{A}\times\vec{B} = A\,B\sin\theta\;\hat{n}$$

$$\sin\theta = \frac{\left|\vec{A}\times\vec{B}\right|}{\left|\vec{A}\right|\left|\vec{B}\right|}$$

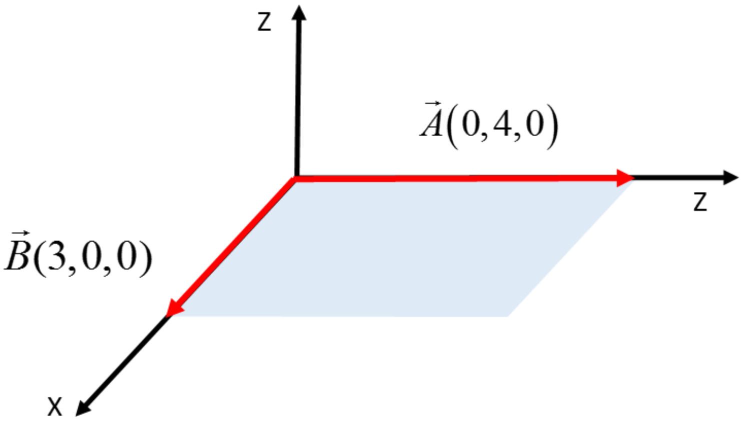
Run the mscript **cemVectorsB.m**

A = [1 0 1] B = [0 1 1]

angle is θ = 1.0472 rad = 60.0000 deg

⇐

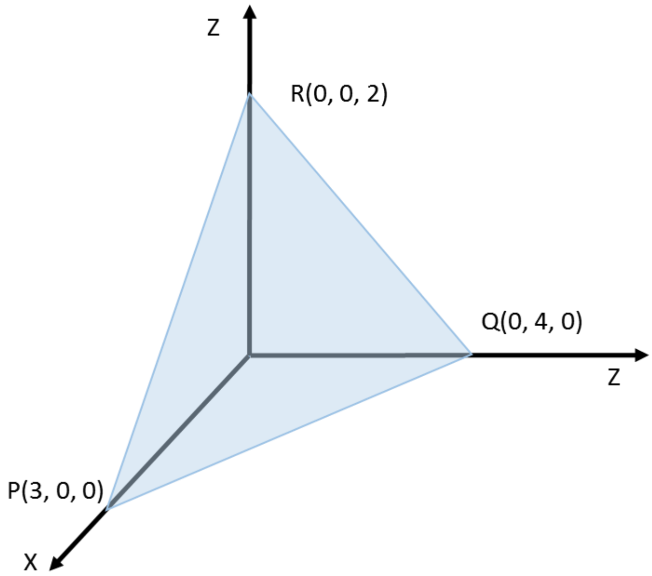
Example Find the components of the unit vector $\hat{n}$ perpendicular to the shaded regions formed by the vectors $\vec{A}$ and $\vec{B}$



$$\vec{C} = \vec{A} \times \vec{B} = -12\hat{k} = \left|\vec{C}\right|\hat{n} \qquad n = \frac{\vec{C}}{\left|\vec{C}\right|} = \frac{-12\hat{k}}{12} = -\hat{k}$$

$$n_x = 0 \quad n_y = 0 \quad n_z = -1$$

Example Find the components of the unit vector $\hat{n}$ perpendicular to the shaded regions formed by the points R, P, Q.

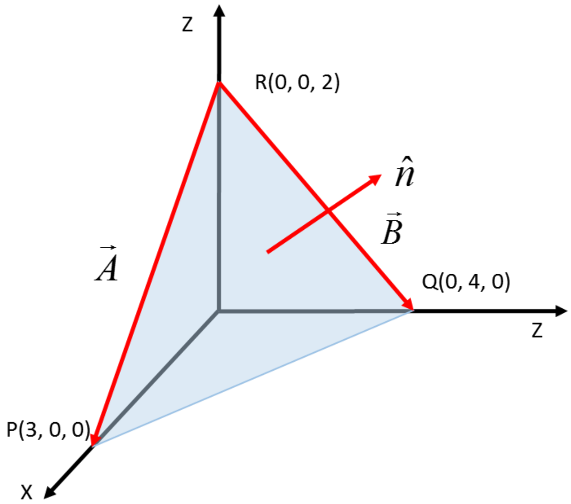


Let $\vec{A}$ be the vector pointing from R to P and $\vec{B}$ be the vector pointing from R to Q. Then the vectors are $\vec{A}(3,0,-2)$ and $\vec{B}(0,4,-2)$.

$$\vec{A}\times\vec{B} = A\,B\sin\theta\;\hat{n}$$

Matlab Command Window

```
A = [3 0 -2]    B = [0 4 -2]
C = cross(A,B)       C = [8   6   12]
Cmag = norm(C)   Cmag = 15.6205
n = C./Cmag
n = [0.5121   0.3841   0.7682]
```



The Cartesian components of the vector $\hat{n}$ are (0.5121, 0.3841, 0.7682)

TRANSFORMATION OF COORDINATES DUE TO ROTATION

What is the change in the components of a vector due to a rotation of the coordinate system from X Y Z to X ’Y ’Z’ ?

The transformation matrix $\mathbf{R}$ due to a rotation uses the following notation

$$X \,\&\, X' \to 1 \quad Y \,\&\, Y' \to 2 \quad Z \,\&\, Z' \to 3$$

θ_{11} is the angle between axes X' and X

θ_{32} is the angle between axes Z' and Y

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \quad \text{where} \quad R_{nm} = \cos\left(\theta_{mn}\right) \qquad m = 1,2,3 \quad n = 1,2,3$$

$\vec{V}\left(V_x, V_y, V_z\right)$ in the XYZ coordinate system

$\vec{V}'\left(V'_x, V'_y, V'_z\right)$ in the X’Y’Z’ coordinate system

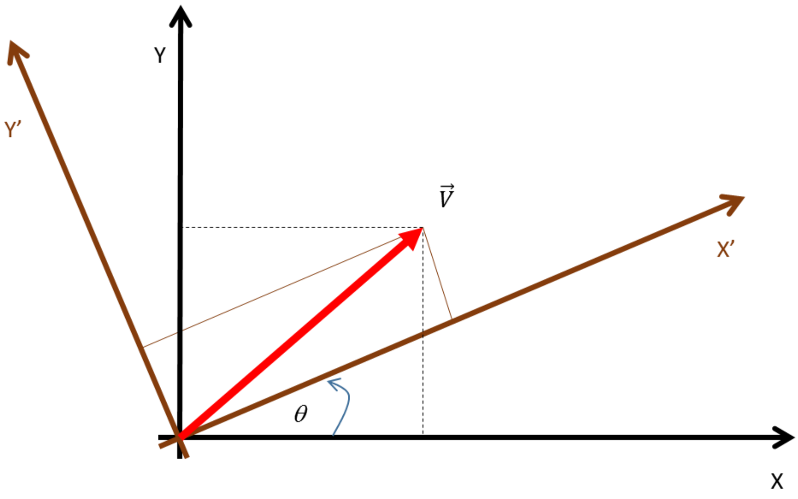
$$\vec{V}' = \mathbf{R}\vec{V}^T \quad \text{where} \quad \vec{V}^T = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

The mscript **cemVectorsC.m** can be modified to calculate the rotation matrix and the components of the vector in the X’Y’Z’ frame of reference.

Rotation of the XY axes around the Z axis

Consider the vector $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$ in the unprimed frame of reference.

What will be its components in the primed frame of reference that is rotated by an angle θ in an anticlockwise direction in the XY plane?



Angles between the axes XYZ and X'Y'Z'

$$\theta_{11} = \theta \qquad \theta_{12} = 90^o - \theta \qquad \theta_{13} = 90^o$$
$$\theta_{21} = \theta + 90^o \qquad \theta_{22} = \theta \qquad \theta_{23} = 90^o$$
$$\theta_{31} = 90^o \qquad \theta_{32} = 90^o \qquad \theta_{33} = 0^o$$

Transformation of vector components

$$\left(V'_x, V'_y, V'_z\right) = \begin{pmatrix} \cos(\theta) & \cos(90^o - \theta) & \cos(90^o) \\ \cos(\theta + 90^o) & \cos(\theta) & \cos(90^o) \\ \cos(90^o) & \cos(90^o) & \cos(0^o) \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$\left(V'_x, V'_y, V'_z\right) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$V'_x = \cos(\theta) V_x + \sin(\theta) V_y$$
$$V'_y = -\sin(\theta) V_x + \cos(\theta) V_y$$
$$V'_z = V_z$$

Example

V(2, 3, 0) in XYZ frame of reference

XYZ rotated by 30^o anticlockwise in XY plane to give the X'Y'Z' frame

What are the components of V in the X'Y'Z' frame?

⇒

Run the mscript **cemVectorsC.m**

Inputs: V = [2 3 0] theta = 30

Output displayed in Command Window: Vdash = [3.2321 1.5981 0]

⇐