DOING PHYSICS WITH PYTHON

NONLINEAR [1D] DYNAMICAL SYSTEMS
FIXED POINTS, STABILITY ANALYSIS, BIFURCATIONS

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Google drive

GitHub

cs100.py   cs101.py   cs103.py   cs104.py   cs10.5.py

INTRODUCTION

To review many aspects of the behaviour of nonlinear systems, we will consider a number of examples of the solutions for nonlinear ordinary differential equation of the form

$$\dot{x} = f(x) \qquad \dot{x} \equiv dx/dt$$

The system will be in equilibrium at a fixed-point x_e where

$$\dot{x} = 0 \qquad f(x_e) = 0$$

When $x = x_e$, $f(x_e) = 0$ then x_e often called a **steady state solution**.

To analyse the stability, consider a small perturbation $e(t)$ from an equilibrium position

$$x(t) = x_e(t) + e(t)$$

From a Taylor expansion, it can be shown that

$$e(t) = e(0)\, e^{f'(x_e)t}$$

If $f'(x_e) > 0$ then $e(t)$ grows exponentially and if $f'(x_e) < 0$, then $e(t)$ decays exponentially to zero.

Thus, the stability of a fixed point is determined from the function $f'(x_e)$ $\left(f'(x) \equiv df / dx \right)$

Stable fixed point $f'(x_e) < 0$ where $x \to x_e$

Marginally stable fixed point $f'(x_e) = 0$ where $x \to x_e$ or $x \to \pm\infty$

Unstable fixed point $f'(x_e) > 0$ where $x \to \pm\infty$

The ODEs are solved using the Python function **odeint.** To reproduce the following plots, you need to change simulation parameters and comment/uncomment parts of the code.

Bifurcation means a structural change in the orbit of a system when a parameter is changed. The point where the bifurcation occurs is known as the **bifurcation point**. The orbit and the fixed point may change dramatically at bifurcation points as the character of an attractor or a repellor are altered. A graph of the parameter values versus the fixed points of the system is known as a **bifurcation diagram**.

The [1D] nonlinear system's ODE can be expressed as

$$\dot{x}(t) = f\left(x(t), r\right)$$

and the fixed points of the system are

$$f\left(x_e(t), r\right) = 0$$

where r is the bifurcation parameter. So, the fixed points and their stability depends upon the bifurcation parameter.

Using a number of examples, three important bifurcations, namely the **saddle node**, **pitchfork**, and **transcritical** bifurcations are discussed. for [1D] systems.

Example 1 **SADDLE NODE BIFURCATION** **cs_100.py**

$\dot{x}(t) = r + x(t)^2$ r is an adjustable constant

$$f(x) = r + x^2 \quad f'(x) = 2x$$

$$\dot{x} = 0 \;\Rightarrow\; x_e = 0 \text{ and } x_e = \pm\sqrt{-r}$$

Thus, there are three possible fixed points;

$r > 0$ no fixed points

$r = 0$ one fixed point $x_e = 0$

$r < 0$ two fixed points $x_e = -\sqrt{-r}$ $x_e = +\sqrt{-r}$

The system's behaviour can be considered in terms of the **velocity vector field**. The system vector field is represented by a vector for the velocity at each position x. The arrow for the velocity vector at point x is to the right (+X direction) if $\dot{x} > 0$ and to the left (-X direction) if $\dot{x} < 0$. So, the flow is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. At the points where $\dot{x} = 0$, there are no flows and such points are called **fixed points**.

$r > 0$ there are no fixed-points

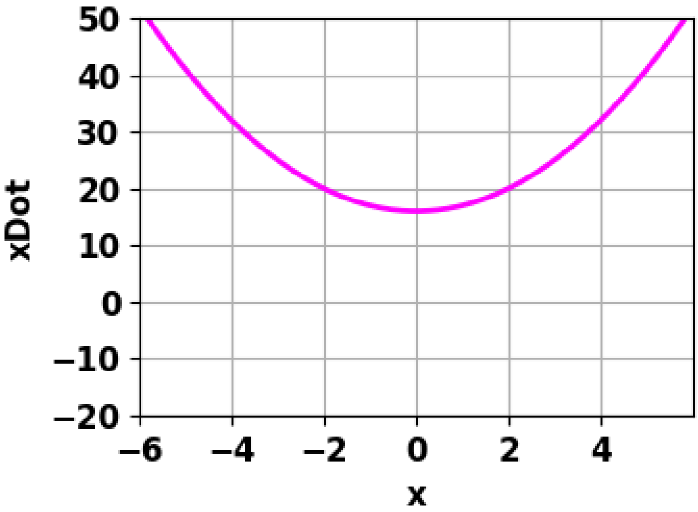


Fig. 1.1 If $r > 0$ then there are no fixed points

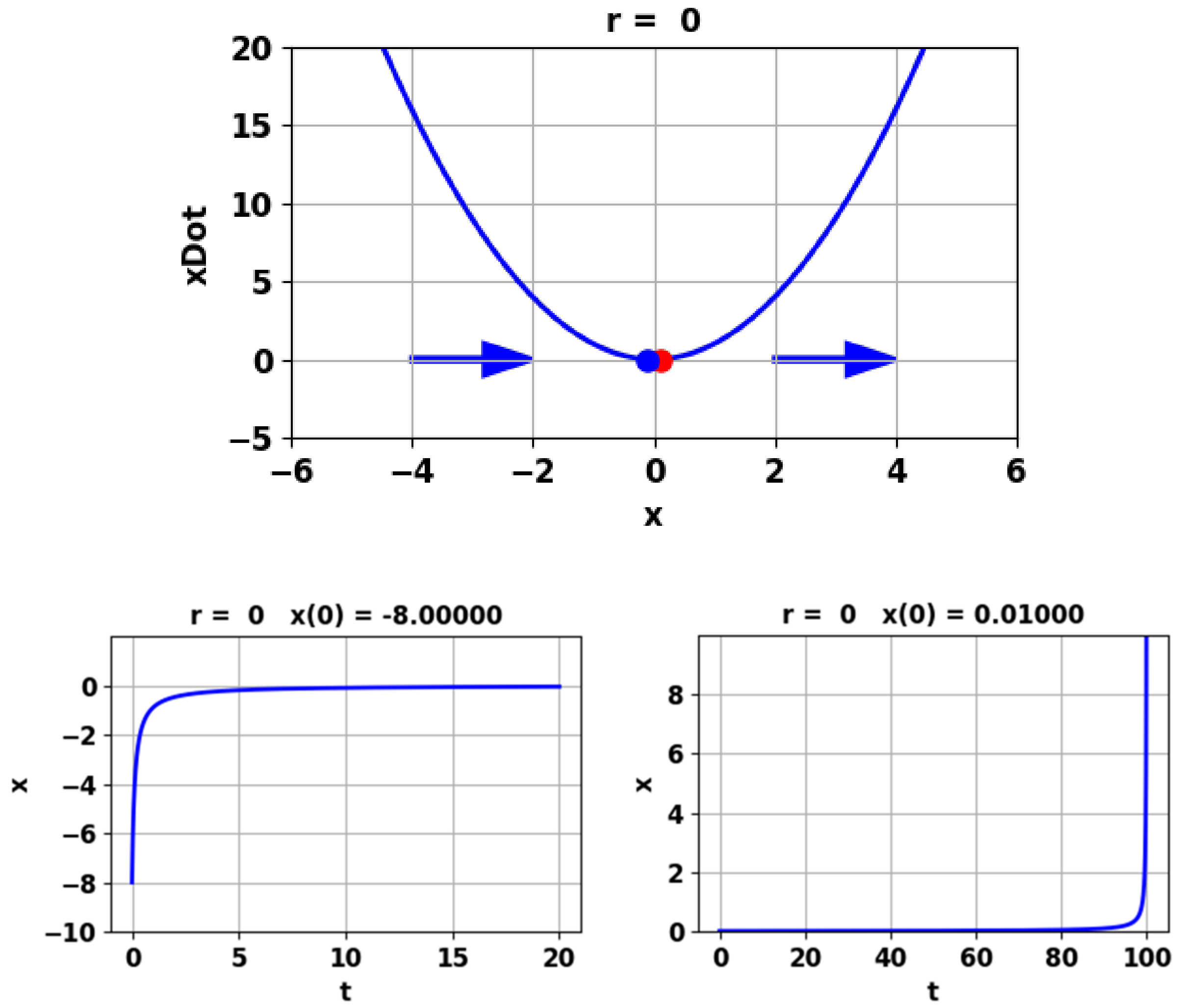
$r = 0$

$$r = 0 \quad \dot{x} = x^2 \quad x_e = 0 \quad f\,'(x_e = 0) = 0$$

$$x(0) = 0 \quad \dot{x}(t) = 0 \quad \Rightarrow t \to \infty \quad x \to 0$$

$$x(0) < 0 \quad \dot{x}(t) > 0 \quad \Rightarrow t \to \infty \quad x \to 0$$

$$x(0) > 0 \quad \dot{x}(t) > 0 \quad \Rightarrow t \to \infty \quad x \to +\infty$$



Fig, 1.2 Fixed point: $r = 0$, $x_e = 0$.

Blue **dot** is a stable fixed point (negative slope)

Red **dot** is an unstable fixed point (positive slope).

$r < 0$

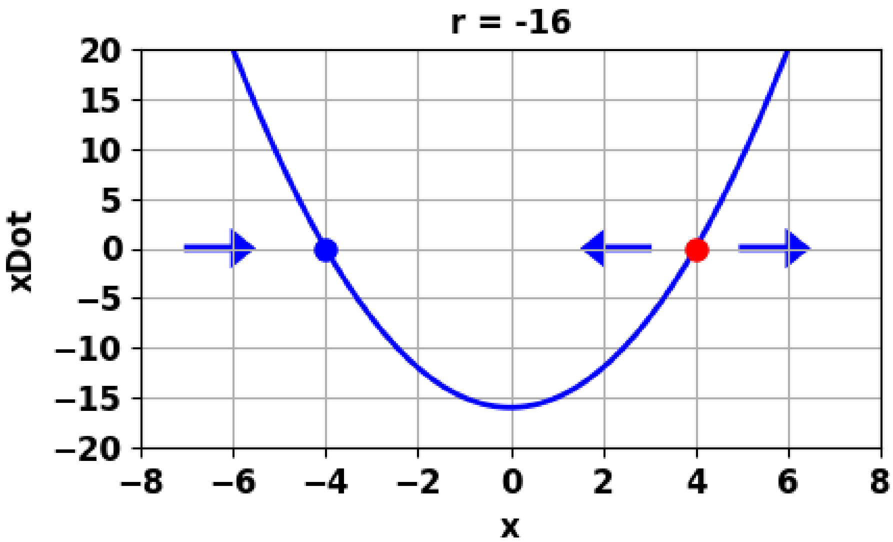
There are two fixed points

$$\dot{x} = r\,x - x^2 \qquad f(x) = r\,x - x^2 \qquad f\,'(x) = 2\,x$$

$$x_e = -\sqrt{-r} \qquad f\,'(x_e) < 0 \quad \Rightarrow \quad \text{stable}$$

$$x_e = +\sqrt{-r} \qquad f\,'(x_e) > 0 \quad \Rightarrow \quad \text{unstable}$$

Let r = -16 then the two fixed points are x_e = -4 (stable) and x_e = +4 (unstable).



This is a very simple system but its dynamics is highly interesting. The bifurcation in the dynamics occurred at $r = 0$ (bifurcation point), since the vector fields for $r < 0$ and $r > 0$ qualitatively different.

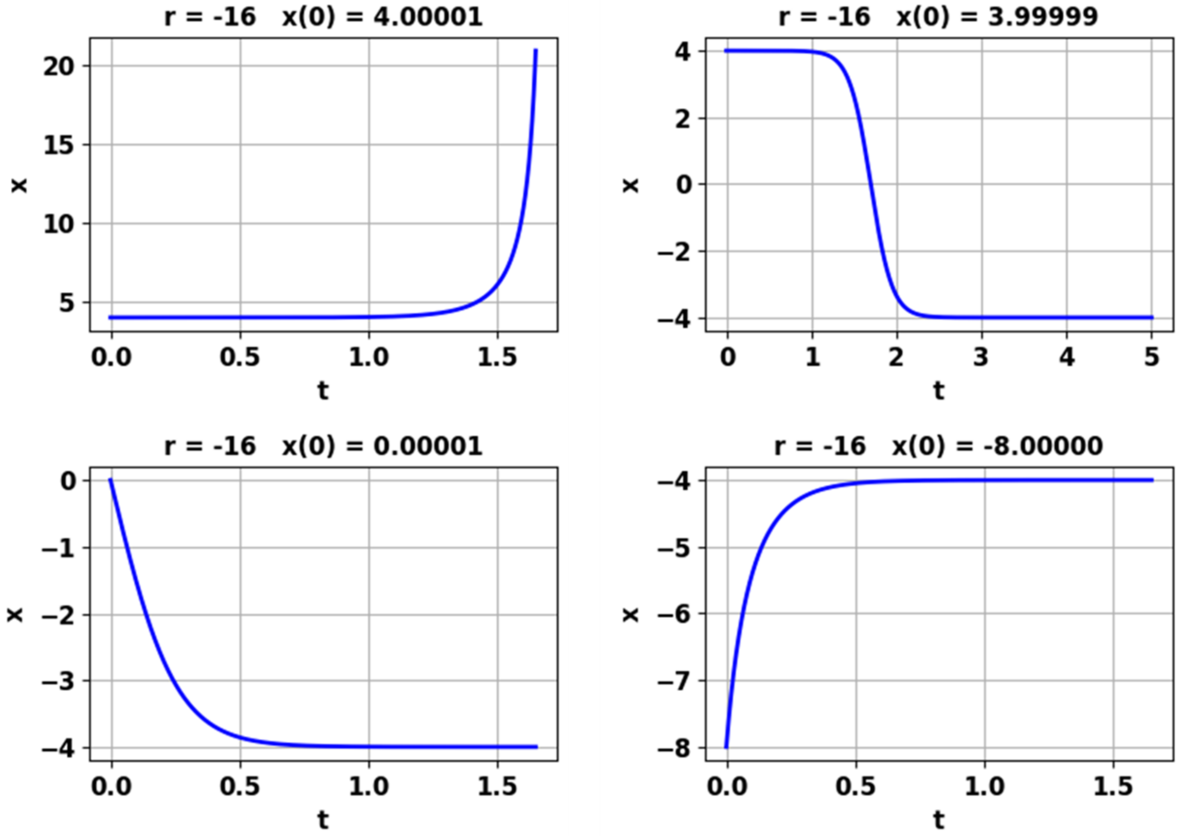


Fig. 1.3 Stable fixed point x_e = -4 (blue dot, negative slope)

Unstable fixed point x_e = +4 (red dot, positive slope)

$$x(0) > 4 \quad t \to \infty \quad \Rightarrow \quad x(t) \to \infty$$

$$x(0) < 4 \quad t \to \infty \quad \Rightarrow \quad x(t) \to -4$$

Figure 1.4 shows the **bifurcation diagram** for the fixed points x_e as a function of the **bifurcation parameter *r***.

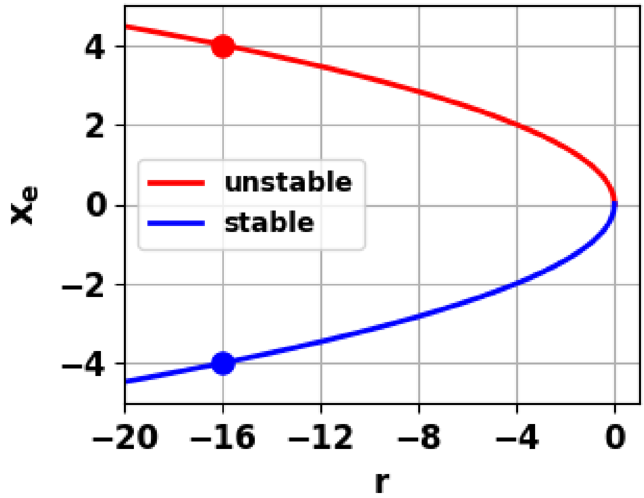


Fig. 1.4 Saddle node bifurcation diagram. The two fixed points for $r < 0$ merge as r goes to zero.

This is an example of a **subcritical saddle node bifurcation** since the fixed points exist for values of the parameter below the bifurcation point $r < 0$.

If we were to consider the system $\dot{x} = r - x^2$ than this would be an example of a **supercritical saddle node bifurcation**, since the equilibrium points exist for values of above the bifurcation point $r = 0 \left(r > 0 \;\Rightarrow\; x_e = \pm\sqrt{r} \right)$.

Example 2 Transcritical bifurcation cs_101.py

The **transcritical bifurcation** is one type of bifurcation in which the stability characteristics of the fixed points are changed for varying values of the parameters.

$\dot{x}(t) = r\,x(t) - x(t)^2$ $\quad$ r is an adjustable constant

$$f(x) = r\,x - x^2 \qquad f\,'(x) = r - 2\,x$$

$\dot{x} = 0 \;\Rightarrow\; x_e = 0$ and $x_e = 0,\, x_e = r \quad f\,'(r) = -r$

This shows that for $r = 0$ the system has only one equilibrium point at $x = 0$. For $r \neq 0$, there are two distinct equilibrium, $x_e = 0$ and $x_e = r$.

If $r > 0$, $f\,'(r) = -r < 0$ and the equilibrium point origin is stable (a sink).

If $r < 0$, $f\,'(r) = -r > 0$ and the equilibrium point origin is unstable (a source).

$$r = 0 \quad \dot{x} = -x^2 \quad x_e = 0$$

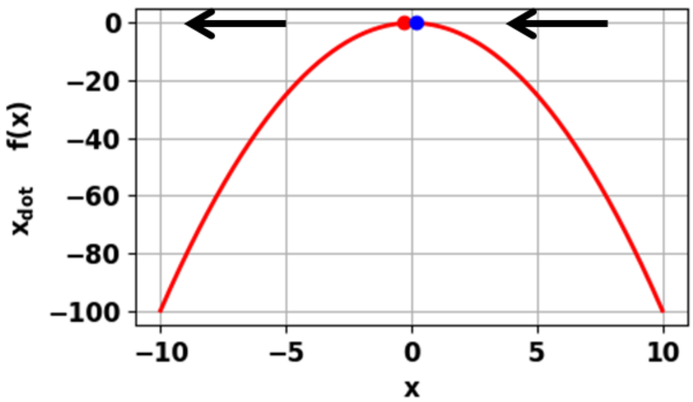
$$f\,'(x_e = 0) = 0$$

$$x(0) < 0 \quad \dot{x}(t) < 0$$

$$\Rightarrow t \to \infty \quad x \to -\infty$$

$$x(0) > 0 \quad \dot{x}(t) < 0$$

$$\Rightarrow t \to \infty \quad x \to 0$$



$$r < 0 \quad \dot{x} = r\,x - x^2$$

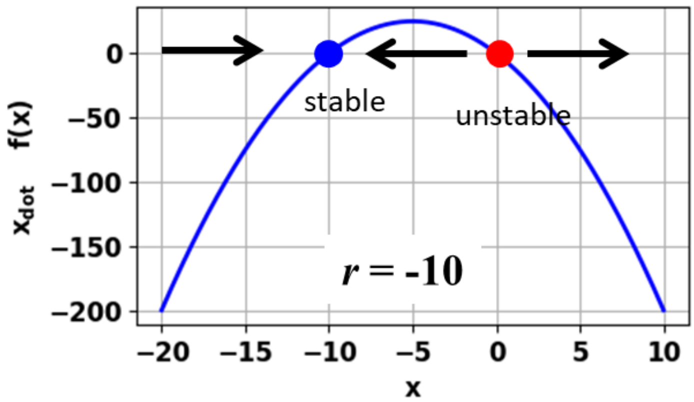
$$f\,'(x) = r - 2x$$

$$\dot{x} = 0 \quad x_e = 0 \quad f\,'(0) < 0$$

$$\dot{x} = 0 \quad x_e = r$$

$$f\,'(x_e) = r - 2x_e$$

$$f\,'(r) = -r > 0$$



$$r > 0$$

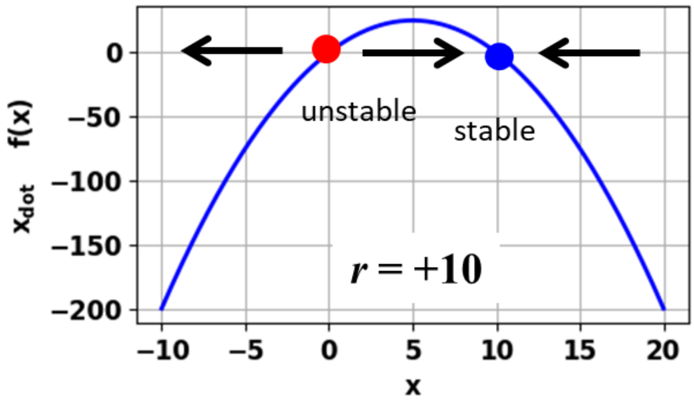
$$\dot{x} = 0 \quad x_e = 0$$

$$f\,'(0) = r > 0$$

$$\dot{x} = 0 \quad x_e = r$$

$$f\,'(x_e) = r - 2x_e$$

$$f\,'(r) = -r < 0$$



This type of bifurcation diagram is known as **transcritical bifurcation**. In this bifurcation, an exchange of stabilities has taken place between the two fixed points of the system.

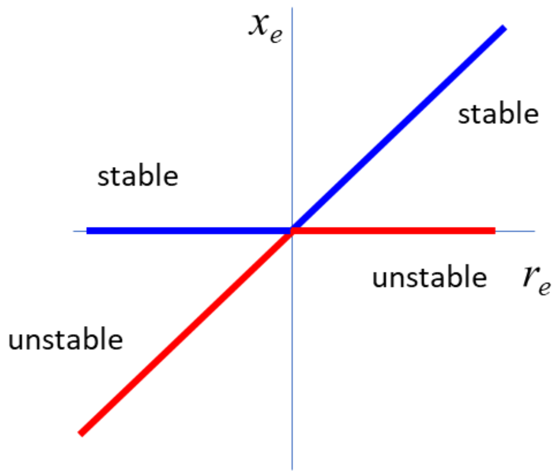


Fig. 2.1

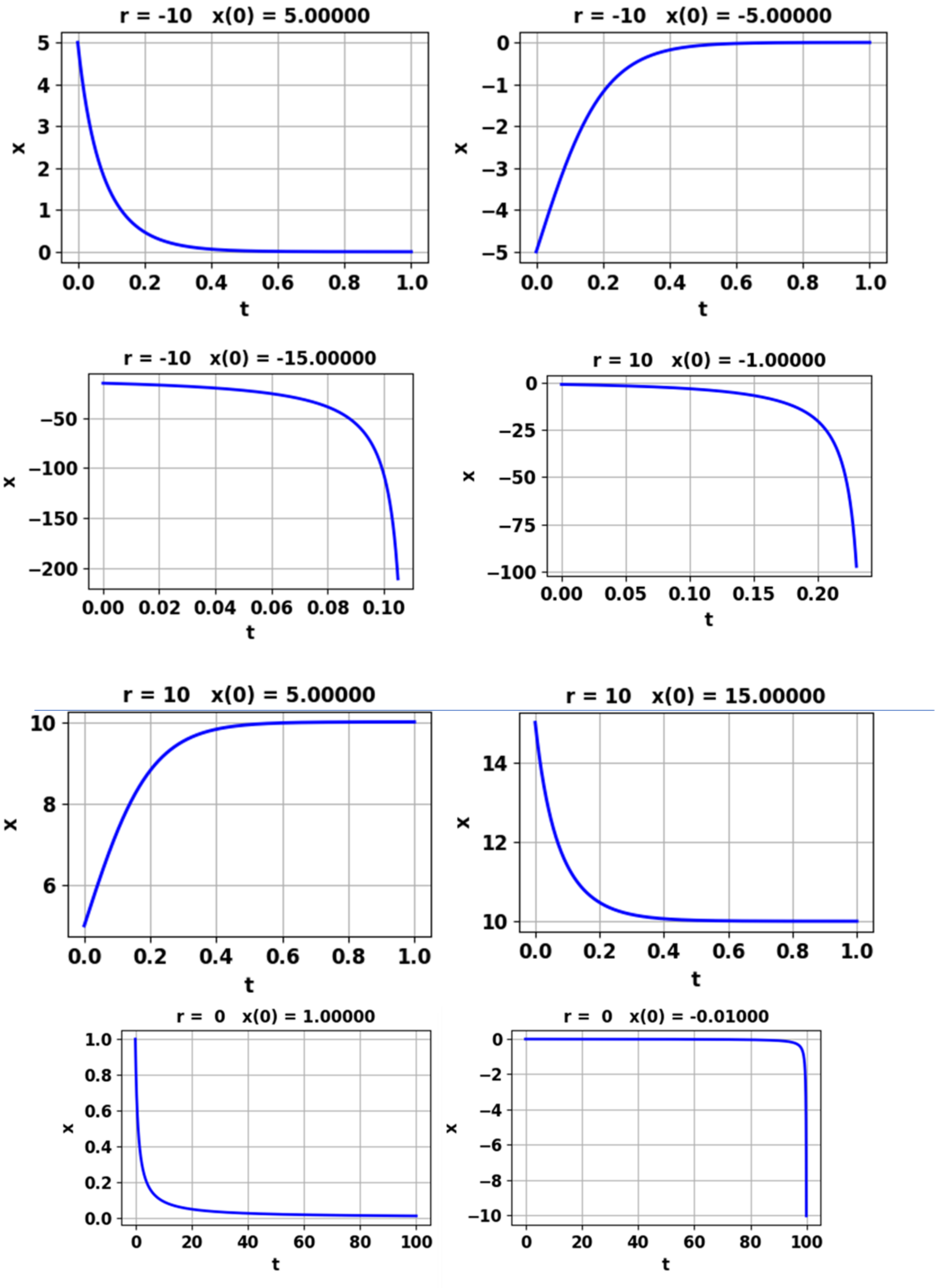


Fig. 2.2 Time evolution plots for $r = 0$, $r = -10$ and $r = +10$.

Example 3 Pitchfork bifurcation cs_103.py

A pitchfork bifurcation in a one-dimensional system appears when the system has symmetry between left and right directions. In such a system, the fixed points tend to appear and disappear in symmetrical pair.

$\dot{x}(t) = r\,x(t) - x(t)^3$ r is an adjustable constant

$f(x,r) = r\,x - x^3 \quad f\,'(x,r) = r - 3x^2$

The system is invariant under the transformation

$$x \to -x \quad r(-x) - (-x)^3 = -\left(rx - x^3\right) = -\dot{x}$$

Fixed points of the system:

r < 0 one fixed point

$$\dot{x} = 0 \Rightarrow \; x_e = 0 \quad f\,'(0) = r < 0 \quad \text{stable}$$

***r* = 0** one fixed point

$$\dot{x} = 0 \Rightarrow \; x_e = 0 \quad f\,'(0) = 0 \quad \text{marginally stable}$$

$$x(0) < 0 \quad \dot{x}(0) > 0 \quad t \to \infty \quad x(t) \to -\infty$$

$$x(0) > 0 \quad \dot{x}(0) < 0 \quad t \to \infty \quad x(t) \to 0$$

***r* > 0** three fixed points

$$\dot{x} = 0 \quad x_e = 0 \quad f\,'(0) = r > 0 \quad \text{unstable}$$

$$\dot{x} = 0 \quad x_e = \pm\sqrt{r} \quad f\,'(\pm\sqrt{r}) = -2r < 0 \quad \text{stable}$$

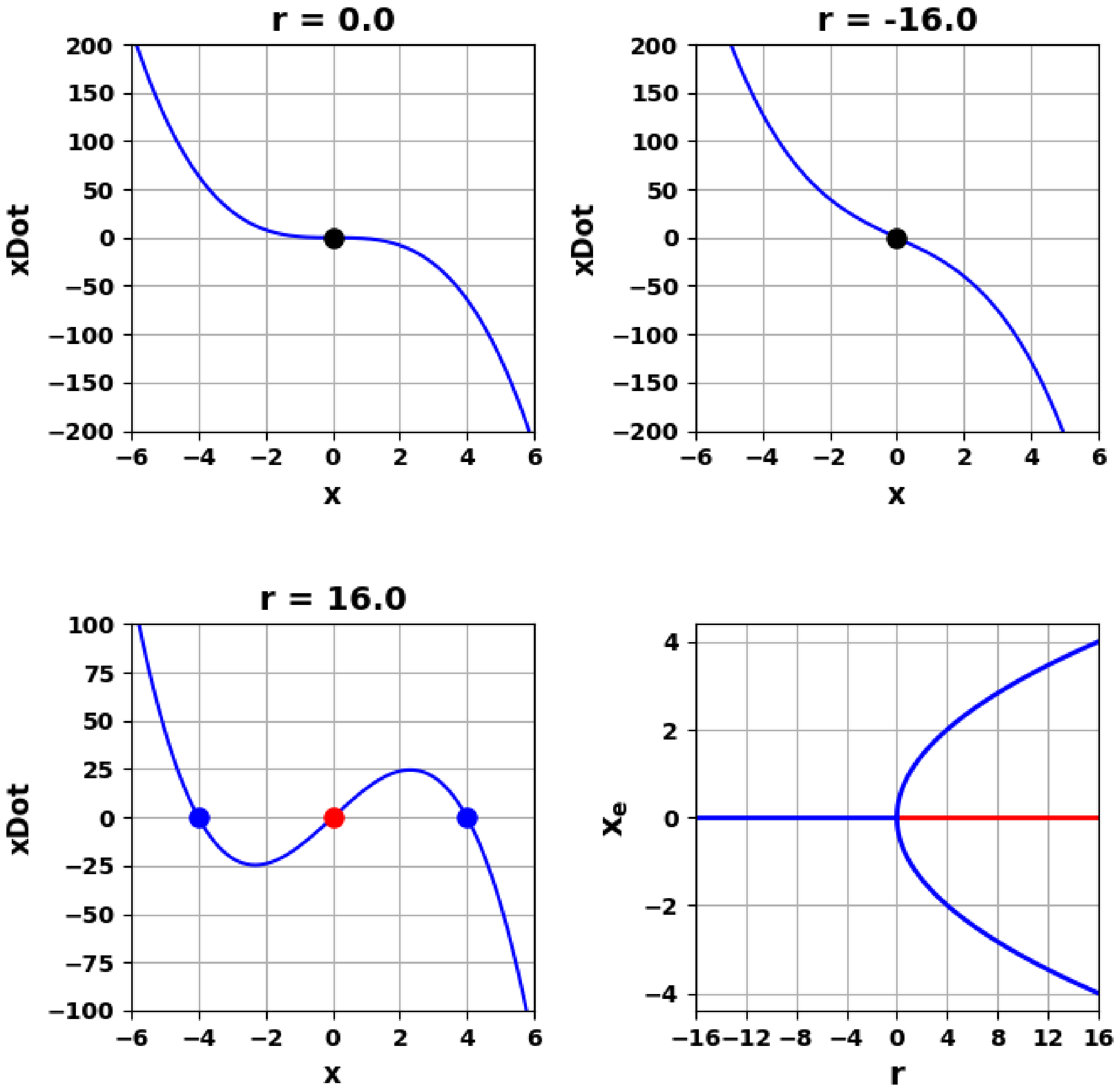


Fig. 3.1

$r = 0$ one fixed point: $x_e = 0$ stable

$r = -16$ one fixed point: $x_e = 0$ stable

$r = +16$ three fixed points: $x_e = 0$ unstable

$x_e = -4$ stable,

$x_e = +4$ stable

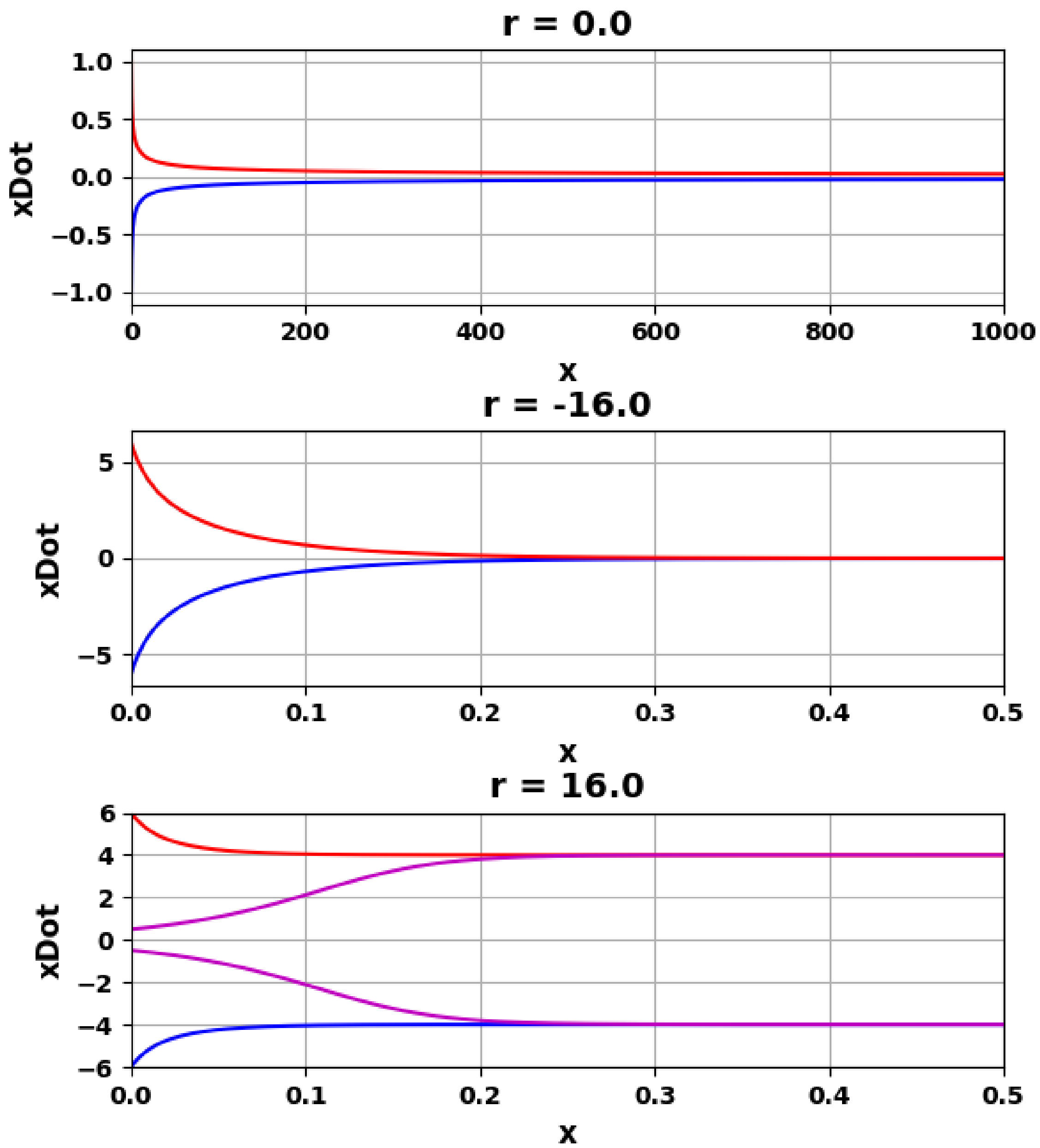


Fig. 3.2 **Supercritical pitchfork bifurcation**

$r = 0$ one fixed point: $x_e = 0$ stable

$r = -16$ one fixed point: $x_e = 0$ stable

$r = + 16$ three fixed points: $x_e = 0$ unstable

$x_e = - 4$ stable

$x_e = + 4$ stable

The pitchfork bifurcations occur when one fixed point becomes three at the bifurcation point Pitchfork bifurcations are usually associated with the physical phenomena called symmetry breaking. For the **supercritical pitchfork bifurcation**, the stability of the original fixed point changes from stable to unstable and a new pair of stable fixed points are created above and below the bifurcation point.

From the pitchfork-shape bifurcation diagram, the name ‘pitchfork’ becomes clear. But it is basically a pitchfork trifurcation of the system. The bifurcation for this vector field is called a supercritical pitchfork bifurcation, in which a stable equilibrium bifurcates into two stable equilibria.

The transformation $x \to -x$, gives the subcritical pitchfork bifurcation $\left(\ddot{x} = rx + x^3\right)$ as shown in the following example.

Example 4 Subcritical pitchfork bifurcation cs_104.py

$\dot{x}(t) = r\,x(t) + x(t)^3$ r is an adjustable constant

$$f(x) = r\,x + x^3 \quad f'(x) = r + 3x^2$$

$r < 0$ three fixed points

$\dot{x} = 0 \quad x_e = 0 \quad f'(0) = r < 0$ stable

$\dot{x} = 0 \quad x_e = \pm\sqrt{-r} \quad f'(\pm\sqrt{-r}) = 2r < 0$

$r = 0$ one fixed point

$\dot{x} = 0 \Rightarrow x_e = 0 \quad f'(0) = 0$ marginally stable

$x(0) < 0 \quad \dot{x}(0) < 0 \quad t \to \infty \quad x(t) \to -\infty$

$x(0) > 0 \quad \dot{x}(0) > 0 \quad t \to \infty \quad x(t) \to +\infty$

$r > 0$ one fixed point

$\dot{x} = 0 \Rightarrow x_e = 0 \quad f'(0) = r > 0$ unstable

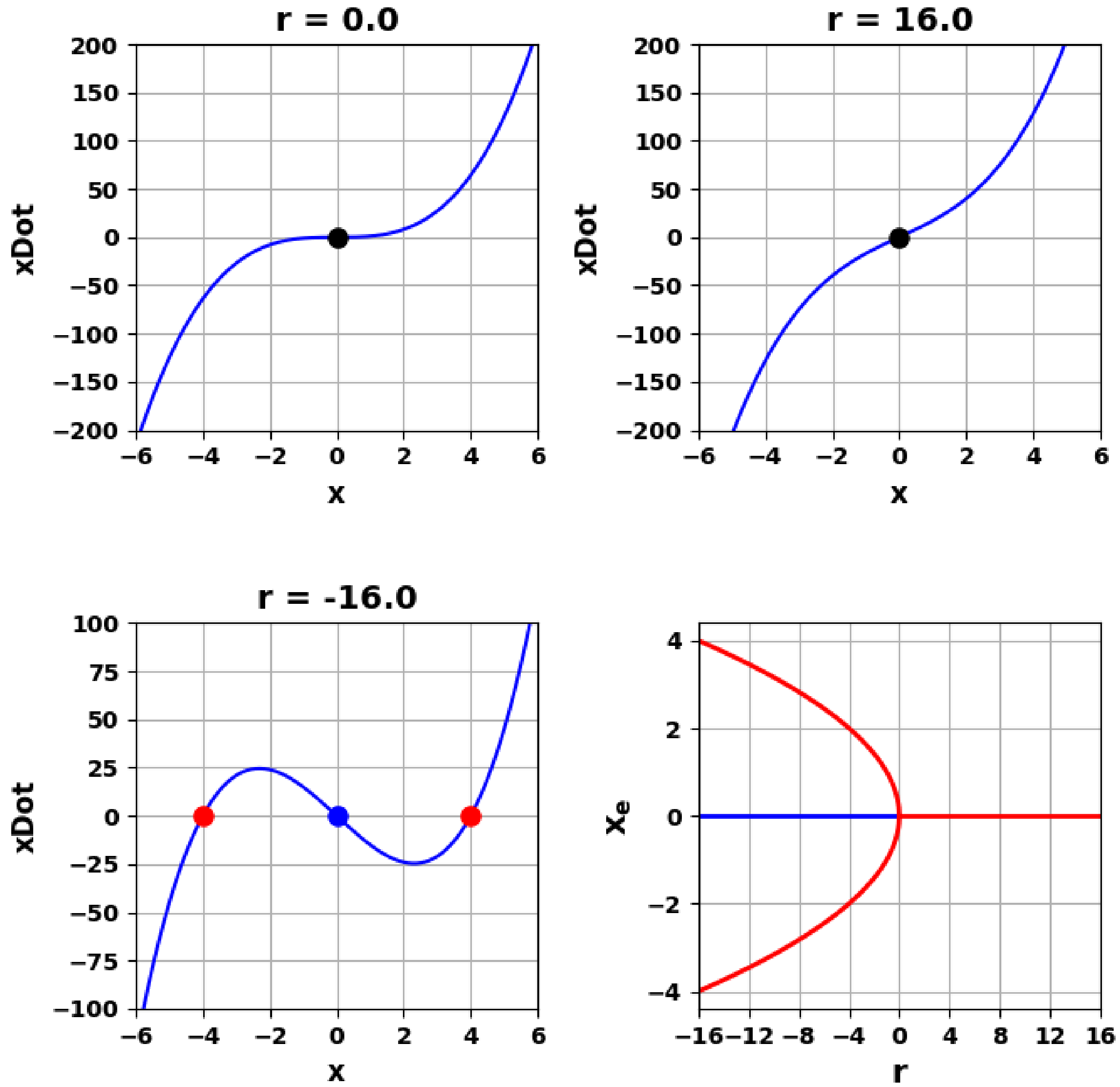


Fig. 4.1 **Subcritical bifurcation**

In a **subcritical bifurcation**, the stability of the original fixed point again changes from stable to unstable but a new pair of now unstable fixed points are created at the bifurcation point.

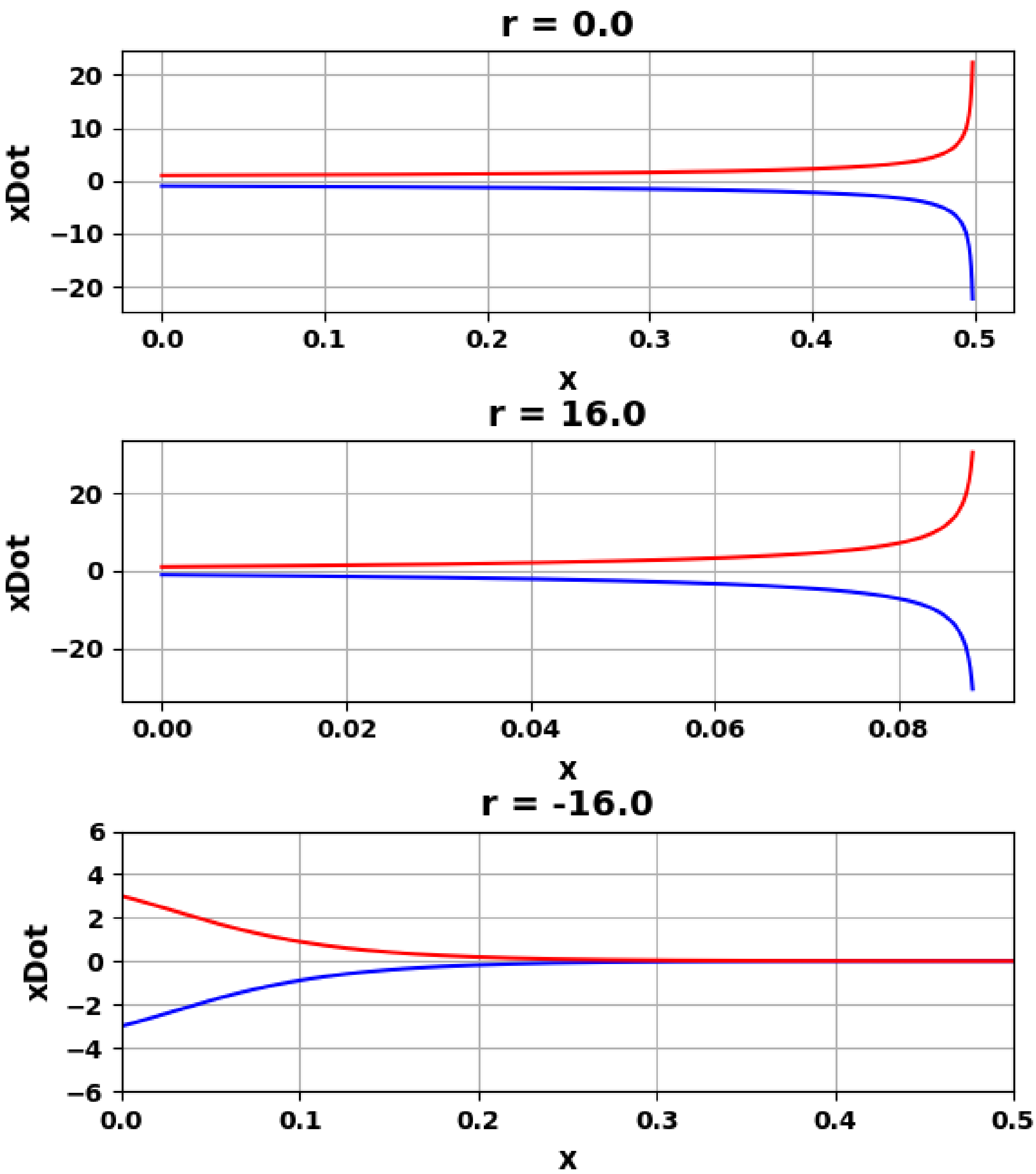


Fig. 4.2 Fixed points

$x_e = 0$ is unstable for $r \geq 0$ $\leftarrow x_e \rightarrow$

x_e is unstable for $r \geq 0$ $\leftarrow x_e \rightarrow$

x_e is stable for $r < 0$ $\rightarrow x_e \leftarrow$

Example 5 $\dot{x}(t) = r\,x(t) + x(t)^3 - x(t)^5$

cs_105.py

$\dot{x} = r\,x + x^3 - x^5$ r is an adjustable constant

$f(x) = r\,x + x^3 - x^5 \quad f'(x) = r + 3x^2 - 5x^4$

$$\dot{x} = 0 \quad \Rightarrow \quad x_e\left(r + {x_e}^2 - {x_e}^4\right) = 0$$

$$x_e = 0 \quad -{x_e}^4 + {x_e}^2 + r = 0$$

$$+z^2 - z - r = 0 \qquad z = {x_e}^2$$

$$z = \tfrac{1}{2}\left(1 \pm \sqrt{1+4r}\right)$$

$$x_e = \pm\sqrt{\tfrac{1}{2}\left(1 \pm \sqrt{1+4r}\right)}$$

$$f'(x_e) = r + 3{x_e}^2 - 5{x_e}^4$$

The bifurcation diagram shown in Fig. 5.1. has in addition to a subcritical pitchfork bifurcation at the origin, two symmetric saddle node bifurcations that occur when $r = -1/4$. We can imagine what happens to the solution x(t) as r increases from negative values, assuming there is some noise in the system so that $x(t)$ fluctuates around a stable fixed point. For $r < -1/4$, the solution $x(t)$ fluctuates around the stable fixed point $x_e = 0$. As r increases into the range $-1/4 < r < 0$, the solution will remain close to the stable fixed point $x_e = 0$. However, a catastrophic event occurs as soon as $r > 0$. The fixed point $x_e = 0$ is lost and the solution will jump up or down to one of the fixed points. A similar catastrophe can happen as r decreases from positive values. In this case, the jump occurs as soon as $r < -1/4$ Since the behaviour of $x(t)$ is different depending on whether we increase or decrease r, we say that the system exhibits **hysteresis**.

The existence of a subcritical pitchfork bifurcation can be very dangerous in engineering applications since a small change in the physical parameters of a problem can result in a large change in the equilibrium state. Physically, this can result in the collapse of a structure.

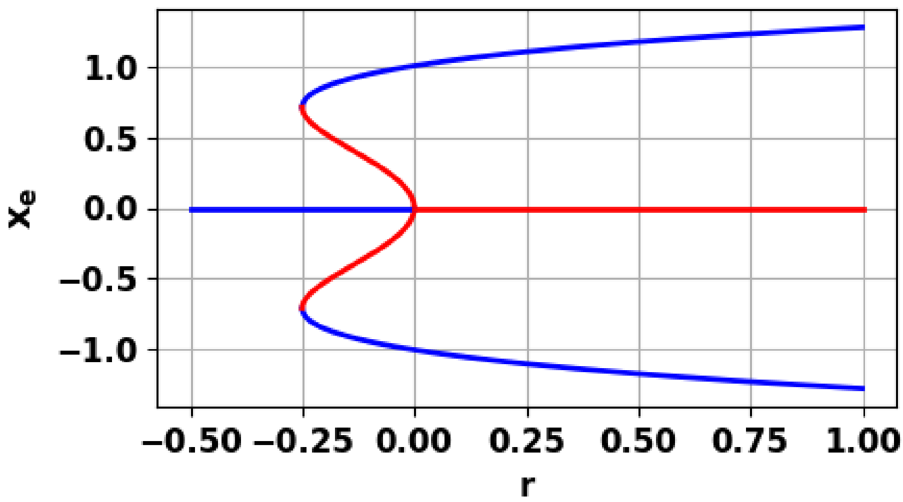
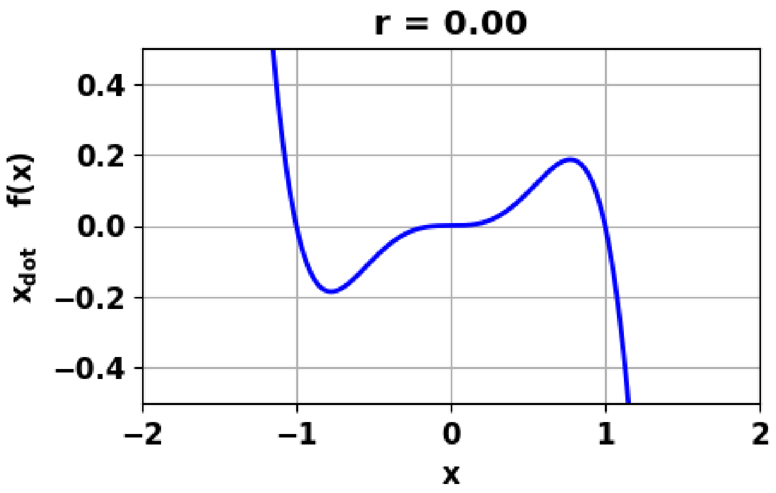
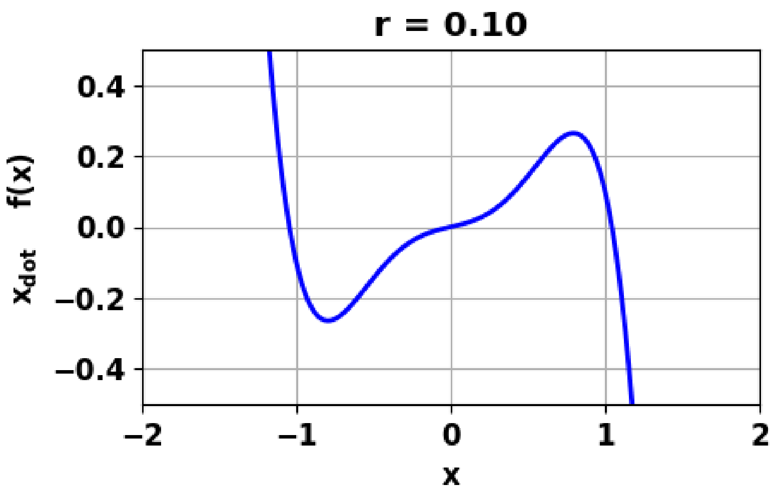
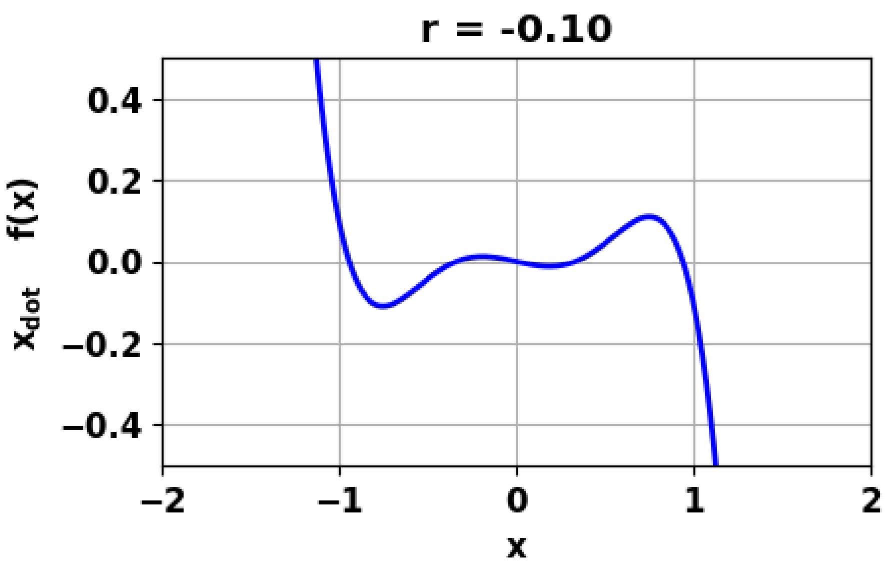


Fig. 5.1 Subcritical pitchfork bifurcation at the origin, and two symmetric saddle node bifurcations that occur when $r = -1/4$.


r = 0.00
x_{dot} f(x)
0.4
0.2
0.0
−0.2
−0.4
−2
−1
0
1
2
x


r = 0.10
x_{dot} f(x)
0.4
0.2
0.0
−0.2
−0.4
−2
−1
0
1
2
x


r = -0.10
x_{dot} f(x)
0.4
0.2
0.0
−0.2
−0.4
−2
−1
0
1
2
x

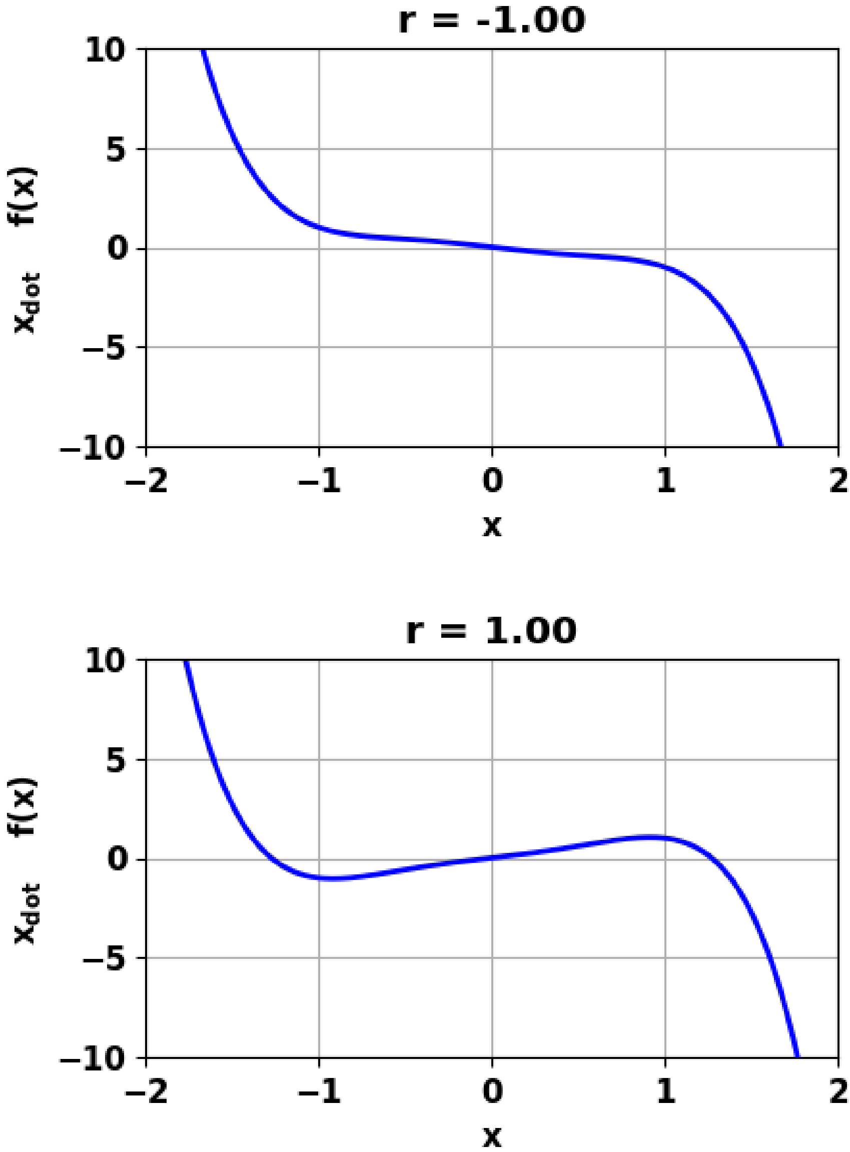


Fig. 5.2 Sequence of plots for the fixed points for a range of *r* values.

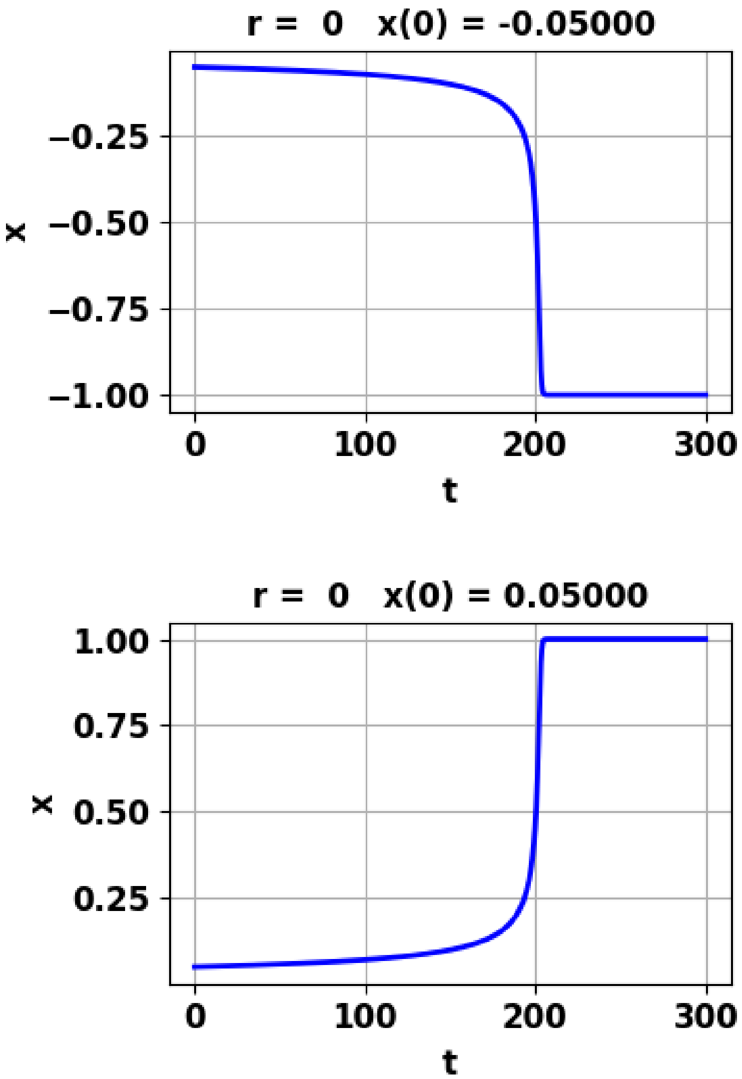


Fig. 5.3 Slight differences in the initial conditions can lead to dramatic differences in the steady state value for *x*.