# DOING PHYSICS WITH MATLAB

# MATHEMATICAL ROUTINES

## COMPUTATION OF ONE-DIMENSIONAL INTEGRALS

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## DOWNLOAD DIRECTORY FOR MATLAB SCRIPTS

**math_integration_1D.m**
Demonstration mscript evaluating the integral of two functions using a number of different methods

**simpson1d.m**
Function to give the integral of a function using Simpson's 1/3 rule.

## NUMERICAL INTEGRATION
## COMPUTATION OF ONE-DIMENSIONAL INTEGRALS

The function **simpson1d.m** is a very versatile , accurate and easy to implement function that can be used to evaluate a definite integral of a function between a lower bound and an upper bound. It is easier to use than the standard Matlab integration functions such as quad. The function **simpson1d.m** is described in detail below.

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We want to compute a number expressing the definite integral of the function $f(x)$ between two specific limits $a$ and $b$

$$I = \int_a^b f(x)\,\mathrm{d}x$$

The evaluation of such integrals is often called ***quadrature***.

We will consider the following integrands to test the accuracy of different integration procedures:

(1) $f(x) = \cos(x)$

(2) $f(x) = \mathrm{e}^{j\,x}$

The integrals are evaluated over a quarter cycle in an attempt to find a minimum number of partitions of the domain that produce accurate results within a reasonable time. For the testing procedures, the following integrals are to be evaluated from $a = 0$ to $b = \pi/2$

$$I = \int_0^{\pi/2} \cos(x)\,\mathrm{d}x \quad \text{and} \quad I = \int_0^{\pi/2} e^{jx}\,\mathrm{d}x$$

These integrals can be evaluated analytically and the exact answers are

$$I = \int_0^{\pi/2} \cos(x)\,\mathrm{d}x = \left[\sin(x)\right]_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1$$

$$I = \int_0^{\pi/2} \mathrm{e}^{jx}\,\mathrm{d}x = \left[\frac{1}{j}\left(\mathrm{e}^{jx}\right)\right]_0^{\pi/2} = \frac{1}{j}\left(\mathrm{e}^{j\pi/2} - 1\right) = 1 + j$$

The numerical procedures assume that neither the integrand $f(x)$ nor any of its derivatives become infinite at any point over the domain of the integration $[a, b]$, and that the limits of integration $a$ and $b$ are finite. Furthermore, we assume that $f(x)$ can be computed, or its values are known at $N$ points $x_c$ where $c = 1, 2, \ldots, N$ that are distributed in some manner over the domain $[a, b]$ with $x_1 = a$ and $x_N = b$.

The qualifier *closed* signifies that the integration involves the values of the integrand at both the end-points. If only one or none of the end points are included, the qualifiers used are *semi-closed* and *open* respectively.

A major problem that arises with non-adaptive methods is that the number $N$ of partitions of the function required to provide a given accuracy is initially unknown. One approach to this problem is to successively double the number of partitions, and compare the results as the number of partitions increase.

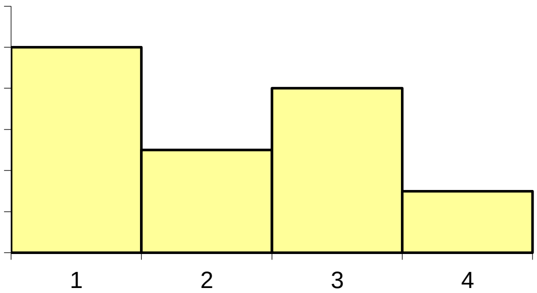
## Closed Rectangle Rule

The region from $a$ to $b$ is divided into $N$ rectangles of equal width $\Delta x \equiv h$ where

$$\Delta x = h = \frac{b-a}{N-1}$$

with each rectangle centre occurring at the points $x_1 = a$, $x_2 = x_1 + h$, … $x_N = b$.

For example, the plot below shows the domain divided into 4 rectangles, $N = 4$. However, the sub-division of the domain by this means extends from $a$ - $h/2$ to $b + h/2$ and not simply from $a$ to b.




The integral is approximated by the contribution from each rectangle, such that

$$I = \sum_{i=1}^{N} f(x_i)\,\Delta x$$

For the case shown in the plot above where $N = 4$, the integral is approximated by

$$I = \{ f(x_1) + f(x_2) + f(x_3) + f(x_4) \} h \quad \text{where } h = \frac{b-a}{3}.$$

## Open Midpoint Rule

The mid-point rule is the first member of a family of open Newton-Cotes rules corresponding to quadratic, cubic and higher-order interpolating polynomials with evenly spaced points. In the open midpoint rule, the area of each rectangle is added to find the integral. The domain *a* to *b* in divided into *N* partitions, with the width of each partition being $h = (b - a)/N$. The function is evaluated at the midpoint $x_i$ of each rectangle where $x_i = a + (h/2)(2N\text{-}1)$. The value of the integral is then given by

$$I = \int_a^b f(x)\,\mathrm{d}x = h\sum_1^N f(x_i)$$

## Trapezoidal Rule

The function $f(x)$ is approximated by a straight line segment connecting adjacent points. The area under the curve is then approximated by adding the area of each trapezium. The interval $a$ to $b$ is divided into $N$-1 partitions of width $h$ ($h \equiv \Delta x$) where $h = (b - a)/(N - 1)$. The area of each partition is simply the area of a trapezium which is its base times its mean height. The area of the $i^{th}$ trapezium is

$$h\left(\frac{f_i + f_{i+1}}{2}\right) \quad \text{where } f_i \equiv f(x_i) \text{ and } i = 1, 2, \ldots N\text{-}1$$

Summing all the trapezia gives the ***composite, closed trapezoidal rule***

$$I = h\left(\frac{f_1 + f_N}{2} + \sum_{2}^{N-1} f_i\right) = h\left(\sum_{1}^{N} f_i - \frac{f_1 + f_N}{2}\right)$$

For evenly distributed spacings, the composite rule is equivalent to the trapezoidal version of the closed ***Newton-cotes rule***.

The Matlab function, trapz implements a procedure to calculate the integral by the trapezoidal rule. For example, the integration of the function $y_1$ w.r.t the variable $x$

```
Integral_1 = trapz(x,y1)     % estimate of the integral
```

## Simpson's 1/3 rule

This rule is based on using a quadratic polynomial approximation to the function *f*(*x*) over a pair of partitions. *N*-1 is the number of partitions where *N* must be **odd** and $h = (b - a) / (N\text{-}1)$. The integral is expressed below and is known as ***composite Simpson's 1/3 rule***.

$$I = \frac{h}{3}\left\{\left( f_1 + f_N + 4\left( f_2 + f_4 + \ldots + f_{N-2}\right) + 2\left( f_3 + f_5 + \ldots + f_{N-1}\right)\right\}$$

Simpson's rule can be written vector form as

$$I = \frac{h}{3}\mathbf{c}\mathbf{f}^{\mathrm{T}}$$

where $\mathbf{c} = \left[1\,4\,2\,4\ldots2\,4\,1\right]$ and $\mathbf{f} = \left[\,f_1\;f_2\;\ldots\;f_N\,\right]$.

Simpson's rule is an example of a ***closed Newton's-Cotes*** formula for integration. Other examples can be obtained by fitting higher degree polynomials through the appropriate number of points. In general we fit a polynomial of degree *N* through *N* +1 points. The resulting polynomials can them be integrated to provide an integration formula. Because of the lurking oscillations associated with the Gibbs effect, higher-order formulas are not used for practical integration.

### simpson1d.m

The function *f* and the lower bound *a* and the upper bound *b* are passed onto the function (in the order *f*, *a*, *b*) and the function returns the value of the integral

```
function integral = simpson1d(f,a,b)

% [1D] integration - Simpson's 1/3 rule
%       f function    a = lower bound    b = upper bound
%       Must have odd number of data points
%       Simpson's coefficients   1 4 2 4 ... 2 4 1

numS = length(f);              % number of data points

sc = 2*ones(numS,1);
sc(2:2:numS-1) = 4;
sc(1) = 1; sc(numS) = 1;

h = (b-a)/(numS-1);

integral = (h/3) * f * sc;
```

# EXAMPLES

(1) $I = \int_0^{\pi/2} \cos(x)\,\mathrm{d}x = 1$ (2) $I = \int_0^{\pi/2} e^{jx}\,\mathrm{d}x = 1 + j$

| Method | Estimate | N |
|---|---|---|
| Closed Rectangle | (1) 1.0950<br>(2) 1.0950 + 1.0950i | 9 |
| | (1) 1.0080<br>(2) 1.0080 + 1.0080i | 99 |
| Open-Midpoint | (1) 0.9978<br>(2) 0.9978 + 0.9978i | 9 |
| | (1) 0.9968<br>(2) 0.9733 + 0.9733i | 99 |
| Trapezoidal | (1) 1.0000<br>(2) 1.0000 + 1.0000i | 9 |
| | (1) 1.0000<br>(2) 1.0000 + 1.0000i | 99 |
| Simpson’s 1/3 | (1) 1.0000<br>(2) 1.0000 + 1.0000i | 9 |

All the integrations took less than one second on a fast Windows computer. Even with only 9 points, the Simpson’s 1/3 rule estimate was equal to the exact values.