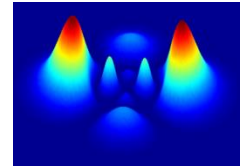# DOING PHYSICS WITH MATLAB

MECHANICS

HORIZONTAL MOTION WITH RESISTANCE

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mec_fr_bv.m
This mscript is used to compute the displacement, velocity and acceleration for the motion of an object acted upon by a resistive force $F_R = -\beta v$. The equation of motion is solved by analytical means (integration of the equation of motion) and by a finite difference numerical method.

mec_fr_bv2.m
This mscript is used to compute the displacement, velocity and acceleration for the motion of an object acted upon by a resistive force $F_R = -\alpha v^2$. The equation of motion is solved by analytical means (integration of the equation of motion) and by a finite difference numerical method.

INTRODUCTION

We will consider the horizontal motion of an object of mass m that is acted upon by a resistive force F_R.
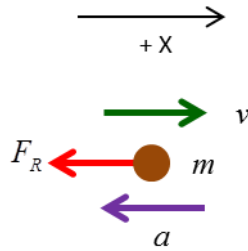
Two very good models for the resistive force F_R are

        Model (1) $F_R = -\beta v$ low speeds

        Model (2) $F_R = -\alpha v^2$ high speeds

where α and β are constants of proportionality.

Model (1) for linear resistance is often applicable when the object is moving with low speeds. In the motion through a fluid, the resistive force $F_R = -\beta v$ is often called the **viscous drag** and it arises from the cohesive forces between the layers of the fluid. The S.I. units for the constant β are N.m^{-1}.s^{-1} or kg.s^{-1}.

Model (2) for quadratic resistance is more applicable for higher speeds. In the motion through fluids, the resistive force $F_R = -\alpha v^2$ is usually called the **drag** and is related to the momentum transfer between the moving object and the fluid it travels through. The S.I. units for the constant are N.m^{-2}.s^{-2} or kg.m^{-1}.

Many problems in the mathematical analysis of particles moving under the influence of resistive forces, you start with the equation of motion. To find velocities and displacements as functions of time you must integrate the equation of motion.

The **equation of motion** for a particle can be derived from Newton's Second Law

        (3) $\vec{a} = \dfrac{1}{m}\sum_i \vec{F}_i$ Newton's Second Law

When the resultant force acting on the object is simply the resistive force, the acceleration a of the object is

        (4) $a = -(\beta/m)v$ Model (1)

        (5) $a = -(\alpha/m)v^2$ Model (2)

MODEL 1 $\quad F_R = -\beta v$

Analytical Approach $\quad F_R = -\beta v$

The force acting on the object is the resistive force $F_R = -\beta v$. In our frame of reference, we will take to the right as the positive direction.

The equation of motion of the object is determined from Newton's Second Law.

$$m a = m \frac{dv}{dt} = F_R = -\beta v \qquad a = -\frac{\beta}{m} v$$

where a is the acceleration of the object at any instance.

The initial conditions are $\quad t = 0 \quad v = v_0 \quad x = 0 \quad a = -(\beta/m) v_0$

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions ($t = 0$ and $v = v_0$) and final conditions (t and v)

$$\frac{dv}{dt} = -\left(\frac{\beta}{m}\right) v$$

$$\frac{dv}{v} = -\left(\frac{\beta}{m}\right) dt$$

$$\int_{v0}^{v} \frac{dv}{v} = \int_{0}^{t} -\left(\frac{\beta}{m}\right) dt$$

$$\left[\log_e (v)\right]_{v_0}^{v} = -\left(\frac{\beta}{m}\right) t$$

$$\log_e \left(\frac{v}{v_0}\right) = -\left(\frac{\beta}{m}\right) t$$

(5) $\qquad v = v_0 \, e^{(-\beta/m)t}$ $\qquad$ **exponential decay**

We can now calculate the displacement x as a function of time t

$$v = \frac{dx}{dt} \quad dx = v \, dt$$

$$v = v_0 \, e^{(-\beta/m)t}$$

$$\int_{0}^{x} dx = \int_{v_0}^{v} v_0 \, e^{(-\beta/m)t} \, dt$$

$$x = -\left(\frac{m v_0}{\beta}\right)\left[e^{(-\beta/m)t}\right]_{0}^{t}$$

(6) $\qquad x = \left(\frac{m v_0}{\beta}\right)\left(1 - e^{(-\beta/m)t}\right)$

The velocity v also can be given as a function of x

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = -\frac{\beta}{m}v$$

$$dv = -\frac{\beta}{m}dx$$

$$\int_{v_0}^{v} dv = \left(-\frac{\beta}{m}\right)\int_{0}^{x} dx$$

(7)
$$x = \left(\frac{\beta}{m}\right)(v_0 - v)$$

$$v = v_0 - \frac{\beta}{m}x$$

straight line graphs

When $v = 0$ the stopping distance is $x_{stopping} = \frac{m v_0}{\beta}$

We can now investigate what happens when $t \rightarrow \infty$

$$t \rightarrow \infty \quad v = v_0\, e^{(-\beta/m)t} \rightarrow 0$$

$$t \rightarrow \infty \quad x = \left(\frac{m v_0}{\beta}\right)\left(1 - e^{(-\beta/m)t}\right) \rightarrow \frac{m v_0}{\beta}$$

The object keeps moving till $v = 0$, which happens only in the limit $t \rightarrow \infty$. Then the object stop at the position $x = \frac{m v_0}{\beta}$.

We can define a **time constant** τ

$$\tau = \frac{\beta}{m}$$

The velocity and displacement can be expressed as

$$v = v_0\, e^{-t/\tau}$$

$$x = \left(\frac{m\, v_0}{\beta}\right)\left(1 - e^{-t/\tau}\right)$$

After a time of about $5\,\tau$, the particle will stop when the speed of the particle becomes zero

$$x_{stopping} = \left(\frac{m\, v_0}{\beta}\right) \qquad \text{stopping time} \sim 5\ \tau$$

The stopping time is independent of the initial velocity but the greater the initial velocity the greater the stopping distance. The larger the constant β, the shorter the stopping distance and quicker it stops and the larger the mass, the greater the stopping distance and it takes a longer time to stop the object.

Numerical Approach $\quad F_R = -\beta\, v$

We can also find the velocity and displacement of the object by solving Newton's Second Law of motion using a finite difference method.

We start with

$$(4) \qquad a = \frac{dv}{dt} = -\left(\beta/m\right)v$$

In the finite difference method we calculate the velocity v and displacement x at N discrete times t_k at fixed time intervals Δt

$$t_1, t_2, \dots , t_k, \dots, t_N \quad \Delta t = t_2 - t_1 \quad t_1 = 0 \quad t_k = (k\text{-}1)\,\Delta t \quad k = 1, 2, 3, \dots , N$$

The acceleration a is approximated by the difference formula

$$\frac{dv\left(t_{k+1}\right)}{dt} \approx \frac{v\left(t_{k+2}\right) - v\left(t_k\right)}{2\,\Delta t}$$

Therefore, the velocity $v\left(t_{k+2}\right)$ at time t_k+2 is

$$\frac{v\left(t_{k+2}\right)-v\left(t_k\right)}{2\Delta t}=-\left(\beta/m\right)v\left(t_{k+1}\right)$$

$$v\left(t_{k+2}\right)=v\left(t_k\right)-\left(2\Delta t\,\beta/m\right)v\left(t_{k+1}\right)$$

Hence, to calculate the velocity $v\left(t_{k+2}\right)$ we need to know the velocity at the two previous time steps t_{k+1} and t_k. We know $t_1 = 0$ and $v(t_1) = v(0) = v_0$.

We estimate the velocity at the second time step t_2

(8) $v(t_2) = v(t_1)+a(t_1)\Delta t = v(t_1)-\left(\beta/m\right)v(t_1)\Delta t$

where we have assumed a constant acceleration in the first time step. We can improve our estimate of $v(t_2)$ by using an average value of the acceleration in the first time step

$$v(t_1)+\left[\frac{a(t_1)+a(t_2)}{2}\right]\Delta t \to v(t_2)$$

$$v(t_1)-\tfrac{1}{2}\left(\beta/m\right)\left[v\left(t_1\right)+v\left(t_2\right)\right]\Delta t \to v(t_2)$$

We can now calculate the velocity $v(t)$ at all times from $t = t_1$ to $t = t_N$.

The acceleration at each time step is

(9) $a\left(t_k\right)=-\left(\dfrac{\beta}{m}\right)v\left(t_k\right)$

The displacement at each time step is

$$v=\frac{dx}{dt}\approx\frac{\Delta x}{\Delta t}$$

$$v\left(t_{k+1}\right)=\frac{x\left(t_{k+2}\right)-x\left(t_k\right)}{2\Delta t}$$

(10) $x\left(t_{k+2}\right)=x\left(t_k\right)+\left(2\Delta t\right)v\left(t_{k+1}\right)$

EXAMPLE $F_R = -\beta v$

The mscript **mec_fr_bv.m** can be used for simulations for the motion of an object acted upon a resistive force of the form $F_R = -\beta v$ (Model 1).
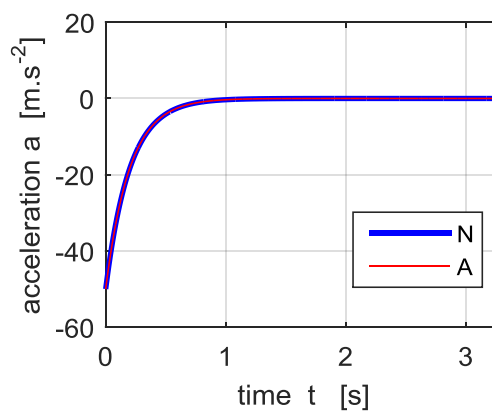
Input parameters

m = 2 kg β = 10 kg.s^{-1} v_0 = 10 m.s^{-1} Δt = 1x10^{-4} s

Outputs   **N** numerical approach   **A** analytical approach

stopping distance $x_{stopping}$ = 2.00 m

time constant τ = 0.200 s 5τ = 1.00 s
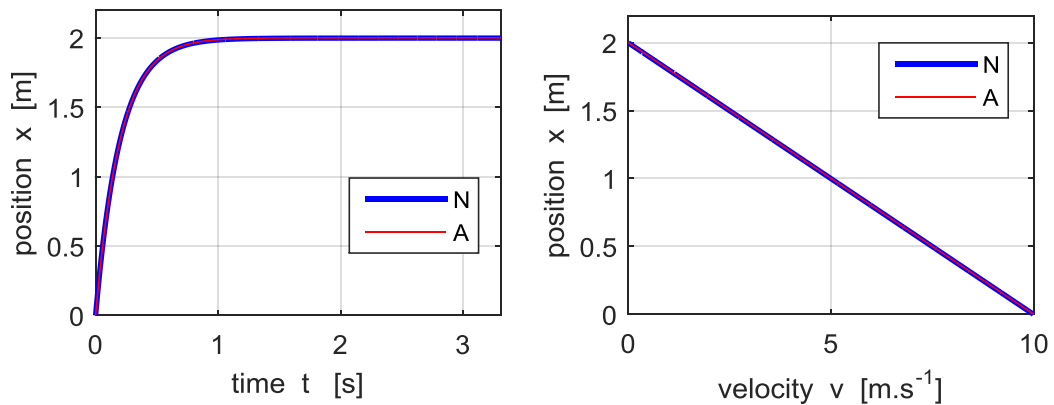
acceleration $t \rightarrow \infty$ $a \rightarrow 0$

velocity $t \rightarrow \infty$ $v \rightarrow 0$
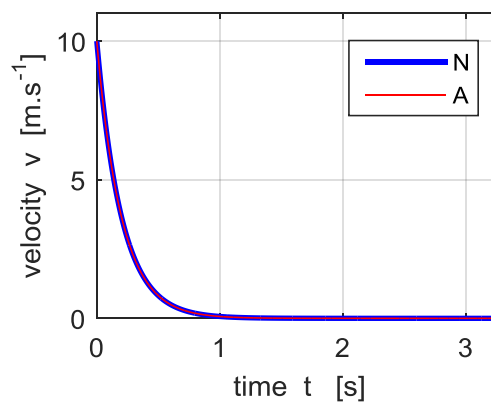
displacement $\quad t \to \infty \quad x \to x_{stopping} = \left(\dfrac{m \, v_0}{\beta} \right)$

For the input parameters used in this simulation, there is excellent agreement between the values calculated using the numerical and analytical approaches.

However, you always need to be careful in using numerical approaches to solve problems. In this instance, you need to check the convergence of results by progressively making the time step Δt smaller.

When the time step is $\Delta t = 1 \times 10^{-2}$ s the numerical and analytical results do not agree. The time step is too large for accurate results using the numerical approach.
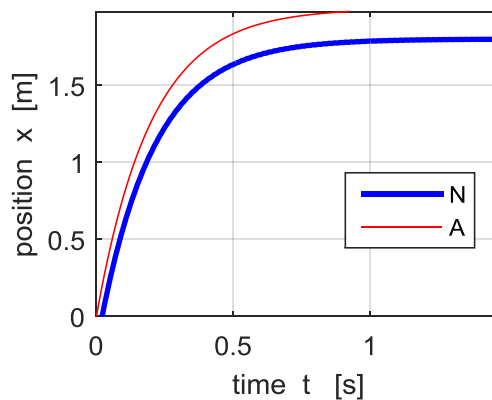
MODEL 2 $F_R = -\alpha v^2$

Analytical Approach $F_R = -\alpha v^2$

The force acting on the object is the resistive force $F_R = -\alpha v^2$. In our frame of reference, we will take to the right as the positive direction.

The equation of motion of the object is determined from Newton's Second Law.

$$m a = m \frac{dv}{dt} = F_R = -\alpha v^2 \qquad a = -\frac{\alpha}{m} v^2$$

where a is the acceleration of the object at any instance.

The initial conditions are $t = 0 \quad v = v_0 \quad x = 0 \quad a = -(\alpha/m) v_0^2$

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions ($t = 0$ and $v = v_0$) and final conditions (t and v)

$$\frac{dv}{dt} = -\left(\frac{\alpha}{m}\right) v^2$$

$$-\left(\frac{\alpha}{m}\right) dt = \frac{dv}{v^2}$$

$$\int_0^t -\left(\frac{\alpha}{m}\right) dt = \int_{v_0}^v \frac{dv}{v^2} =$$

$$-\left(\frac{\alpha}{m}\right) t = \left[-\frac{1}{v}\right]_{v_0}^v = \frac{1}{v_0} - \frac{1}{v}$$

$$\frac{v_0}{v} = 1 + \left(\frac{\alpha v_0}{m}\right) t$$

(11) $$v = \frac{v_0}{1 + \left(\dfrac{\alpha v_0}{m}\right) t}$$

The acceleration is

(12) $$a = -\left(\frac{\alpha v_0}{m}\right)^2 \left(\frac{1}{1 + \left(\dfrac{\alpha v_0}{m}\right) t}\right)^2$$

$$(t \to \infty) \quad a \to -\left(\frac{m}{\alpha}\right)\left(\frac{1}{t^2}\right) \qquad \text{acceleration becomes small very rapidly}$$

We can now calculate the displacement x as a function of time t

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = -(\alpha/m)v^2$$

$$-\left(\frac{\alpha}{m}\right)dx = \frac{dv}{v}$$

$$\int_0^x -\left(\frac{\alpha}{m}\right)dx = \int_{v_0}^v \frac{dv}{v}$$

$$-\left(\frac{\alpha}{m}\right)x = \log\left(\frac{v}{v_0}\right)$$

(13) $$x = \left(\frac{m}{\alpha}\right)\log\left(\frac{v_0}{v}\right)$$

(11) $$v = \frac{v_0}{1 + \left(\dfrac{\alpha\,v_0}{m}\right)t}$$

(14) $$x = \left(\frac{m}{\alpha}\right)\log\left(1 + \left(\frac{\alpha\,v_0}{m}\right)t\right)$$ *surprising result !!!*

As time goes on the displacement gets bigger and bigger. In this simple model, the objects just keeps moving towards the right.

Numerical Approach $F_R = -\alpha v^2$

We can also find the velocity and displacement of the object by solving Newton's Second Law of motion using a finite difference method.

We start with

(4) $$a = \frac{dv}{dt} = -(\alpha/m)v^2$$

In the finite difference method we calculate the velocity v and displacement x at N discrete times t_k at fixed time intervals Δt

$t_1, t_2, \ldots, t_k, \ldots, t_N$ $\Delta t = t_2 - t_1$ $t_1 = 0$ $t_k = (k\text{-}1)\Delta t$ $k = 1, 2, 3, \ldots, N$

The acceleration a is approximated by the difference formula

$$\frac{dv(t_{k+1})}{dt} \approx \frac{v(t_{k+2}) - v(t_k)}{2\Delta t}$$

Therefore, the velocity $v(t_{k+2})$ at time t_k+2 is

$$\frac{v(t_{k+2}) - v(t_k)}{2\Delta t} = -(\varepsilon/m)v(t_{k+1})^2$$

$$v(t_{k+2}) = v(t_k) - (2\Delta t\, \alpha/m)v(t_{k+1})^2$$

Hence, to calculate the velocity $v(t_{k+2})$ we need to know the velocity at the two previous time steps t_{k+1} and t_k. We know $t_1 = 0$ and $v(t_1) = v(0) = v_0$.

We estimate the velocity at the second time step t_2

$$(8) \qquad v(t_2) = v(t_1) + a(t_1)\Delta t = v(t_1) - (\beta/m)v(t_1)\Delta t$$

where we have assumed a constant acceleration in the first time step. We can improve our estimate of $v(t_2)$ by using an average value of the acceleration in the first time step

$$v(t_1) + \left[\frac{a(t_1) + a(t_2)}{2}\right]\Delta t \to v(t_2)$$

$$v(t_1) - \tfrac{1}{2}(\beta/m)\left[v(t_1) + v(t_2)\right]\Delta t \to v(t_2)$$

We can now calculate the velocity $v(t)$ at all times from $t = t_1$ to $t = t_N$.

The acceleration at each time step is

$$(9) \qquad a(t_k) = -\left(\frac{\alpha}{m}\right)v(t_k)^2$$

The displacement at each time step is

$$v = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t}$$

$$v(t_{k+1}) = \frac{x(t_{k+2}) - x(t_k)}{2\Delta t}$$

$$(10) \qquad x(t_{k+2}) = x(t_k) + (2\Delta t)v(t_{k+1})$$

EXAMPLE $F_R = -\alpha v^2$

The mscript **mec_fr_bv2.m** can be used for simulations for the motion of an object acted upon a resistive force of the form $F_R = -\alpha v^2$ (Model 2).
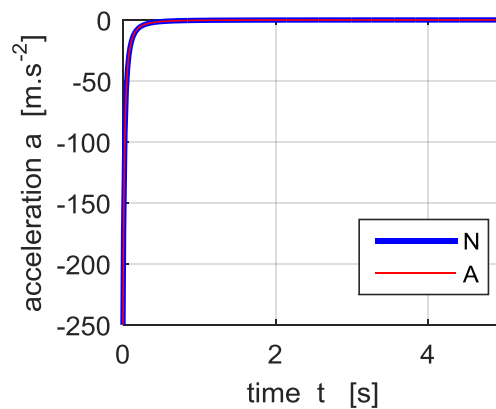
Input parameters

m = 2 kg α = 5 kg.m^{-1} v_0 = 10 m.s^{-1} Δt = 1x10^{-4} s t_{max} = 5 s
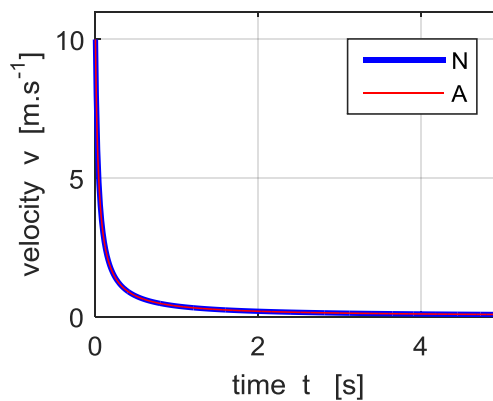
Outputs   **N** numerical approach   **A** analytical approach

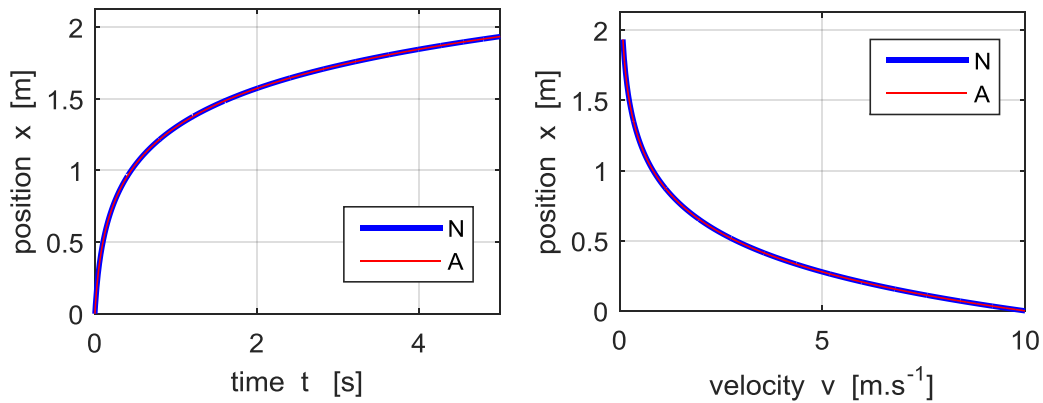acceleration $t \rightarrow \infty$ $a \rightarrow 0$

acceleration quicker gets smaller in magnitude with time

velocity $t \rightarrow \infty$ $v \rightarrow 0$

displacement   does not go to zero as $t \rightarrow \infty$

For the input parameters used in this simulation, there is excellent agreement between the values calculated using the numerical and analytical approaches.

However, you always need to be careful in using numerical approaches to solve problems. In this instance, you need to check the convergence of results by progressively making the time step Δt smaller.

When the time step is $\Delta t = 1 \times 10^{-3}$ s the numerical and analytical results do not agree. The time step is too large for accurate results using the numerical approach. In the numerical approach the position values oscillate about the analytical values.