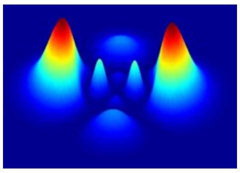
# DOING PHYSICS WITH MATLAB



## MECHANICS

# MOTION OF FALLING OBJECTS WITH RESISTANCE

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## DOWNLOAD DIRECTORY FOR MATLAB SCRIPTS

**mec_fr_mg_bv.m**
Computation of the displacement, velocity and acceleration for the motion of an object acted upon by a resistive force $F_R = -\beta\, v$ and its weight $F_G = m\, g$. The equation of motion is solved by analytical means (integration of the equation of motion) and by a finite difference numerical method.

**mec_fr_mg_bv2.m**
Computation of the displacement, velocity and acceleration for the motion of an object acted upon by a resistive force $F_R = -\alpha\, v^2$ and its weight $F_G = m\, g$. The equation of motion is solved by analytical means (integration of the equation of motion) and by a finite difference numerical method.

**mec_stokes.m**
Computation of the displacement, velocity and acceleration for the motion of an object acted upon by a resistive force $F_R = 6\pi\, \eta_F\, R\, v$ (Stokes Law), the buoyance force $F_{buoy}$ and its weight $F_G = m\, g$. The equation of motion is solved by a finite difference numerical method.

**mec_drag.m**
Computation of the displacement, velocity and acceleration for the motion of an object acted upon by a resistive force $F_R = \frac{1}{2} C_D 6 A \rho_F\; v^2$ and its weight $F_G = m\, g$. The equation of motion is solved by a finite difference numerical method.

**mec_tt_ball.m**
Experimental data for a table tennis ball falling from rest. Used to plot the actual measurements of displacement and time from a video recording with the predicted values of displacement using a finite difference approach.

# INTRODUCTION

We will consider the vertical motion of objects through fluids near the Earth's surface where the acceleration due to gravity is assumed to be constant $g$ = 9.80 m.s$^{-2}$.

The motion of falling objects is usually described with constant acceleration. This is only approximately true. For example, in introductory physics textbooks, two objects of different mass when dropped simultaneously from rest will hit the ground at the same time. This is an idealized situation and ignores the effects of the air resisting the motion of the falling objects. Air resistance, a friction which increases with increasing speed, acts against gravity, so the speed of falling objects tends toward a limit called terminal velocity (terminal speed).

Motion through any real fluid (liquid or gas) gives rise to forces resisting the motion. To a reasonable approximation, fluid resistance tends to depend on either the first power of the speed (a linear resistance) or the second power (a quadratic resistance). Our two models for the resistive force $F_R$ are

| | | |
|---|---|---|
| Model (1) | $F_R = -\beta v$ | low speeds |
| Model (2) | $F_R = -\alpha v^2$ | high speeds |

where $\alpha$ and $\beta$ are constants of proportionality.

**Model (1)** for linear resistance is often applicable when the object is moving with low speeds. In the motion through a fluid, the resistive force $F_R = -\beta v$ is often called the **viscous drag** and it arises from the cohesive forces between the layers of the fluid. The S.I. units for the constant $\beta$ are N.m$^{-1}$.s$^{-1}$ or kg.s$^{-1}$.

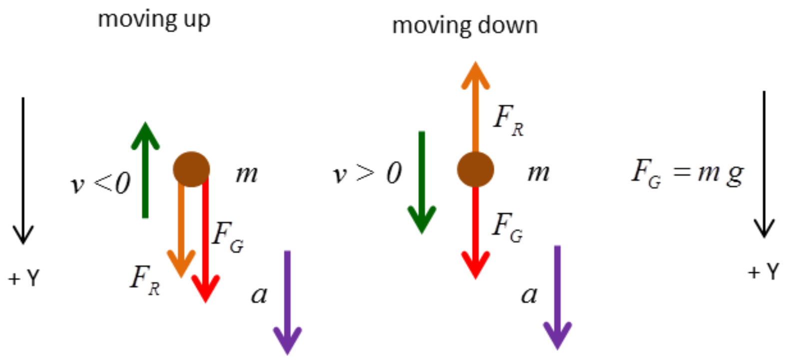
**Model (2)** for quadratic resistance is more applicable for higher speeds. In the motion through fluids, the resistive force $F_R = -\alpha v^2$ is usually called the **drag** and is related to the momentum transfer between the moving object and the fluid it travels through. The S.I. units for the constant are N.m$^{-2}$.s$^{-2}$ or kg.m$^{-1}$.

Many problems in the mathematical analysis of particles moving under the influence of resistive forces, you start with the equation of motion. To find velocities and displacements as functions of time you must integrate the equation of motion.

The **equation of motion** for an object can be derived from Newton's Second Law

$$\vec{a} = \frac{1}{m}\sum_i \vec{F}_i \qquad \text{Newton's Second Law}$$

For the vertical motion of an object through a fluid, the forces acting on the object are the gravitational force $F_G$ (weight) and the resistive force $F_R$. In our frame of reference, we will take down as the positive direction.




Newton's Second Law

$$m\,a = m\,g + F_R \qquad\qquad m\,a = m\,g - F_R$$

Therefore, the vertical acceleration $a$ of an object through a fluid is

$$a = g - (\beta / m)\,v \qquad \text{Model (1)}$$

$$a = g - (\alpha / m)\,v^2\,(v/|v|) \qquad \text{Model (2)}$$

In Model 2

If the object is moving down then $v > 0$

$$a = g - (\alpha / m)\,v^2\,(v/|v|) = g - (\alpha / m)\,v^2\,(+1) = g - (\alpha / m)\,v^2$$

If the object is moving up then $v < 0$

$$a = g - (\alpha / m)\,v^2\,(v/|v|) = g - (\alpha / m)\,v^2\,(-1) = g + (\alpha / m)\,v^2$$

## MODEL 1 $F_R = -\beta v \qquad m a = m g - \beta v$

The mscript used for the plots is **mec_fr_mg_bv.m**

### Analytical Approach $F_R = -\beta v \qquad m a = m g - \beta v$

For the vertical motion of an object through a fluid, the forces acting on the object are the gravitational force $F_G$ (weight) and the resistive force $F_R$. In our frame of reference, we will take down as the positive direction.

The equation of motion of the object is determined from Newton's Second Law.

$$m a = m \frac{dv}{dt} = F_R = mg - \beta v \qquad a = g - \frac{\beta}{m} v$$

where $a$ is the acceleration of the object at any instance.

The initial conditions are $t = 0 \quad v = v_0 \quad x = 0 \quad a = -(\beta / m) v_0$

When $a$ = 0, the velocity is constant $v = v_T$ where $v_T$ is the **terminal velocity**

$$0 = m g - \beta v_T$$

$$v_T = \frac{m g}{\beta}$$ **terminal velocity**

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions ($t$ = 0 and $v = v_0$) and final conditions ($t$ and $v$)

$$a = \frac{dv}{dt} = g - \left(\frac{\beta}{m}\right) v = -\left(\frac{\beta}{m}\right)\left(v - \frac{m g}{\beta}\right)$$

$$u = v - \frac{m g}{\beta} \qquad du = dv \qquad -\left(\frac{\beta}{m}\right) dt = \frac{du}{u}$$

$$-\left(\frac{\beta}{m}\right)\int_0^t dt = \int_{u0}^{u} \frac{du}{u} \qquad -\left(\frac{\beta}{m}\right) t = \left[\log_e(u)\right]_{v_0 - \frac{m g}{\beta}}^{v - \frac{m g}{\beta}} = \log_e\left(\frac{v - \frac{m g}{\beta}}{v_0 - \frac{m g}{\beta}}\right)$$

$$\left(\frac{v - \frac{m g}{\beta}}{v_0 - \frac{m g}{\beta}}\right) = e^{(-\beta / m) t}$$

$$v = \frac{m g}{\beta} + \left(v_0 - \frac{m g}{\beta}\right) e^{(-\beta / m) t} \qquad v_T = \frac{m g}{\beta}$$

Therefore, we can express the velocity $v$ as

$$v = v_T + \left(v_0 - v_T\right) e^{(-\beta/m)\,t} \qquad \textbf{velocity}$$

$$v_0 = 0 \quad \Rightarrow \quad v = v_T\left(1 - e^{(-\beta/m)\,t}\right)$$

$$v_0 = v_T \Rightarrow \quad v = v_T$$

$$v_0 < v_T \Rightarrow \quad v \text{ increases to } v_T$$

$$v_0 > v_T \Rightarrow \quad v \text{ decreases to } v_T$$

In every case, the velocity $v$ tends towards the limiting value $v_T$.

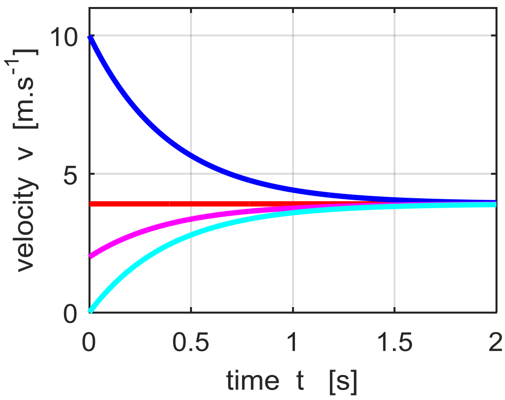
Plots of the velocity $v$ as functions of time $t$

$m$ = 2.00 kg

$\beta$ = 5.00 kg.s$^{-1}$

$g$ = 9.80 m.s$^{-2}$

$v_T$ = 3.92 m.s$^{-1}$




Initial values for velocity $v_0$ [m.s$^{-1}$]

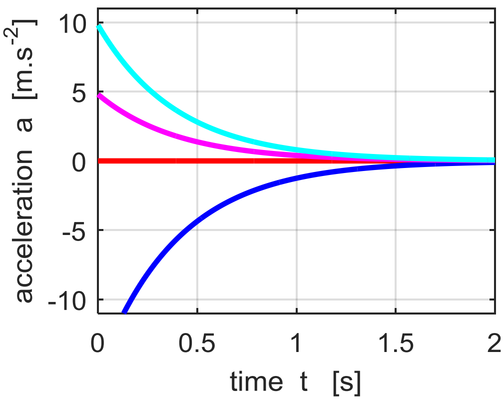
**blue: 10 red: $v_T$ magenta: 2 cyan: 0**

The acceleration $a$ as a function of time $t$ is

$$v = v_T + (v_0 - v_T)e^{(-\beta/m)t}$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(v_T + (v_0 - v_T)e^{(-\beta/m)t}\right)$$

$$a = (v_T - v_0)\left(\frac{\beta}{m}\right)e^{(-\beta/m)t}$$




$$v_0 = 0 \quad \Rightarrow \quad a = \left(\frac{\beta\, v_T}{m}\right)e^{(-\beta/m)t}$$

$$\Rightarrow \quad a = g\, e^{(-\beta/m)t}$$

$$v_0 = v_T \Rightarrow \quad a = 0$$

$$v_0 < v_T \Rightarrow \quad a > 0 \quad \text{and decreases to } 0$$

$$v_0 > v_T \Rightarrow \quad a < 0 \quad \text{and } a \text{ increases to } 0$$

$$t \to \infty \Rightarrow \quad a \to 0$$

Initial values for velocity $v_0$ [m.s$^{-1}$]

**blue: 10 red: $v_T$ magenta: 2 cyan: 0**

We can now calculate the displacement $x$ as a function of velocity $t$

$$v = v_T + (v_0 - v_T)e^{(-\beta/m)t}$$

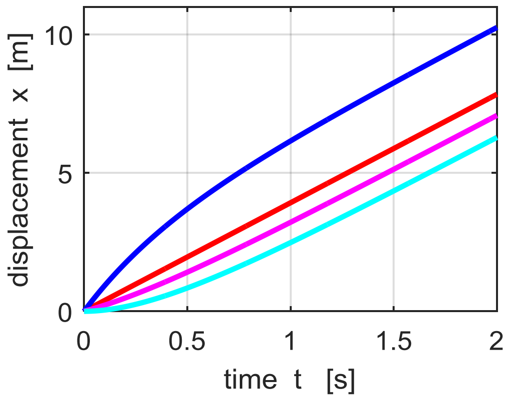
$$v = \frac{dx}{dt} \quad dx = v\,dt$$

$$\int_0^x dx = \int_0^t \left(v_T + (v_0 - v_T)e^{(-\beta/m)t}\right)dt$$

$$x = \left[v_T\,t - \left(\frac{m}{\beta}\right)(v_0 - v_T)e^{(-\beta/m)t}\right]_0^t$$

$$x = v_T\,t - \left(\frac{m}{\beta}\right)(v_0 - v_T)e^{(-\beta/m)t} + \left(\frac{m}{\beta}\right)(v_0 - v_T)$$

$$x = v_T\,t + \left(\frac{m}{\beta}\right)(v_0 - v_T)\left(1 - e^{(-\beta/m)t}\right)$$




$$t \to \infty \qquad x \to \infty$$

$$v_0 = 0 \quad \Rightarrow \quad x = v_T\left(t + \left(\frac{m}{\beta}\right)\left(e^{(-\beta/m)t} - 1\right)\right)$$

Initial values for velocity $v_0$ [m.s$^{-1}$]

**blue: 10 red: $v_T$ magenta: 2 cyan: 0**

So far we have only considered the case where the initial velocity was either zero or a positive quantity ($v_0 \geq 0$), i.e., the object was released from rest or projected downward. We will now consider the case where the object was project vertically upward ($v_0 < 0$). Note: in our frame of reference, the origin is taken as $x = 0$, the position of the object at time $t = 0$; down is the positive direction and up is the negative direction.

When the object is launched upward at time $t = 0$, the initial velocity has a negative value. Let $u$ be the magnitude of the initial velocity $v_0$

$$v_0 < 0 \quad v_0 = -u \quad u > 0$$

Therefore, the equation for the velocity $v$ as a function of time $t$ can be expressed as

$$v = v_T + (v_0 - v_T)e^{(-\beta/m)t}$$

$$v = v_T - (u + v_T)e^{(-\beta/m)t}$$

We can now find the time $t_{up}$ it takes for the object to rise to its maximum height $x_{up}$ above the origin (remember: up is negative). At the highest point $v = 0$, therefore,

$$0 = v_T - (u + v_T)\, e^{(-\beta/m)\,t_{up}}$$

$$t_{up} = \left(\frac{m}{\beta}\right) \log_e\left(1 + \frac{u}{v_T}\right)$$

The maximum height $x_{up}$ reached by the object in time $t = t_{up}$ is

$$x = v_T\, t + \left(\frac{m}{\beta}\right)(v_0 - v_T)\left(1 - e^{(-\beta/m)\,t}\right)$$

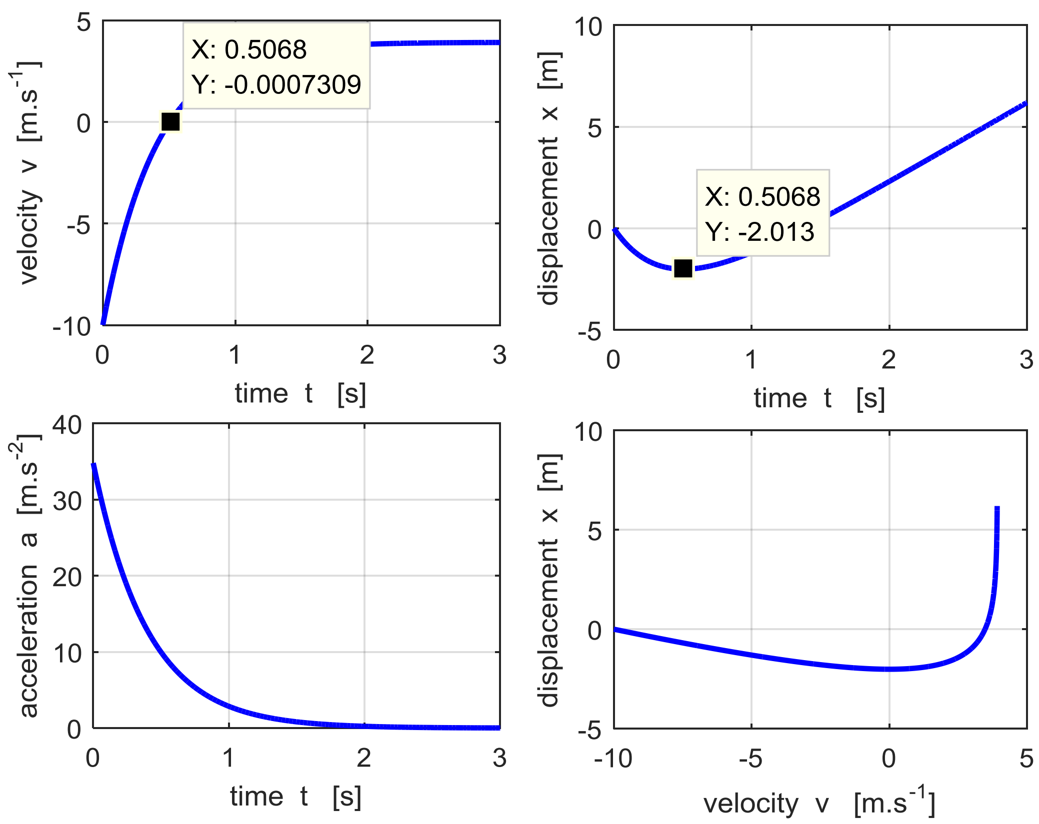
$$x_{up} = v_T\, t_{up} - \left(\frac{m}{\beta}\right)(u + v_T)\left(1 - e^{(-\beta/m)\,t_{up}}\right)$$

For the parameters

$m = 2.00$ kg  $\beta = 5.00$ kg.s$^{-1}$  $g = 9.8$ m.s$^{-2}$  $u = 10$ m.s$^{-1}$  $v_T = 3.92$ m.s$^{-1}$

The time to reach maximum height is $t_{up} = 0.507$ s

The max height $h_{up}$ reached is $h_{up} = 2.013$ m  $x_{up} = -\,2.013$ m



We can find the displacement $x$ as a function of velocity $v$

$$a = v\frac{dv}{dx} = g - (\beta / m)v \qquad dx = \frac{v\,dv}{g - (\beta / m)v}$$

$$\left(\frac{1}{\beta / m}\right)dx = \frac{v\,dv}{(mg / \beta) - v} \qquad v_T = mg / \beta \qquad (\beta / m)\,dx = \frac{v\,dv}{v_T - v}$$

We can integrate this equation by a substitution method or an algebraic manipulation method.

## Substitution Method

$$u = v_T - v \quad du = -dv \quad v = v_T - u \quad dv = -du \quad v_0 = v_T - u_0 \quad u_0 = v_T - v_0$$

$$(\beta / m)\,dx = \frac{-(v_T - u)}{u}du$$

$$\int_0^x (\beta / m)\,dx = \int_{u_0}^{u} \frac{-(v_T - u)}{u}du = \int_{u0}^{u}\left(1 - \frac{v_T}{u}\right)du$$

$$(\beta / m)\,x = \left[u - v_T \log_e(u)\right]_{u_0}^{u} = (u - u_0) - v_T \log_e\left(\frac{u}{u_0}\right)$$

$$(\beta / m)\,x = (v_0 - v) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)$$

$$x = \left(\frac{m}{b}\right)\left((v_0 - v) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

## Algebraic manipulation

$$(\beta / m)\,dx = \frac{v\,dv}{v_T - v}$$

$$\frac{v\,dv}{v_T - v} = v_T\left(\frac{-1}{v_T} + \frac{1}{v_T - v}\right)$$

$$\int_0^x (\beta / m)\,dx = v_T\left(\int_{v_0}^{vu}\left(\frac{-1}{v_T} + \frac{1}{v_T - v}\right)dv\right)$$

$$(\beta / m)\,x = v_T\left[\frac{-v}{v_T} - \log_e(v_T - v)\right]_{v_0}^{v}$$

$$(\beta / m)\,x = \left((v_0 - v) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

$$x = \left(\frac{m}{b}\right)\left((v_0 - v) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

*QED*

For the object projected up with an initial velocity $v_0 = -u$ where $u > 0$, the maximum height reached $x_{up}$ occurs when $v = 0$

$$x = \left(\frac{m}{b}\right)\left((v_0 - v) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

$$x_{up} = \left(\frac{m}{b}\right)\left((-u) + v_T \log_e\left(\frac{v_T + u}{v_T}\right)\right)$$

$$x_{up} = \left(\frac{m}{b}\right)\left(v_T \log_e\left(1 + \frac{u}{v_T}\right) - u\right)$$

Note: up is negative and down is positive in our frame of reference.

## Numerical Approach $F_R = -\beta v \qquad m a = m g - \beta v$

We can also find the velocity and displacement of the object by solving Newton's Second Law of motion using a finite difference method.

We start with

$$a = \frac{dv}{dt} = g - (\beta / m) v \qquad v_T = \frac{m g}{\beta}$$

In the finite difference method we calculate the velocity $v$ and displacement $x$ at $N$ discrete times $t_k$ at fixed time intervals $\Delta t$

$$t_1, t_2, \ldots, t_k, \ldots, t_N \quad \Delta t = t_2 - t_1 \quad t_1 = 0 \quad t_k = (k-1)\,\Delta t \quad k = 1, 2, 3, \ldots, N$$

The acceleration $a$ is approximated by the difference formula

$$\frac{dv(t_{k+1})}{dt} \approx \frac{v(t_{k+2}) - v(t_k)}{2\,\Delta t}$$

Therefore, the velocity $v(t_{k+2})$ at time $t_k$+2 is

$$\frac{v(t_{k+2}) - v(t_k)}{2\,\Delta t} = g - (\beta / m) v(t_{k+1})$$

$$v(t_{k+2}) = v(t_k) + (2\,\Delta t) g - (2\,\Delta t\,\beta / m) v(t_{k+1})$$

$$v(t_{k+2}) = v(t_k) + (2\,\Delta t\, g)\left(1 - \frac{v(t_{k+1})}{v_T}\right)$$

Hence, to calculate the velocity $v(t_{k+2})$ we need to know the velocity at the two previous time steps $t_{k+1}$ and $t_k$ . We know $t_1 = 0$ and $v(t_1) = v(0) = v_0$.

We estimate the velocity at the second time step $t_2$

$$v(t_2) = v(t_1) + a(t_1)\,\Delta t = v(t_1) - (\beta / m) v(t_1)\,\Delta t$$

where we have assumed a constant acceleration in the first time step. We can improve our estimate of $v(t_2)$ by using an average value of the acceleration in the first time step

$$v(t_1) + \left[\frac{a(t_1) + a(t_2)}{2}\right]\Delta t \rightarrow v(t_2)$$

$$v(t_1) - \tfrac{1}{2}(\beta / m)\left[v(t_1) + v(t_2)\right]\Delta t \rightarrow v(t_2)$$

We can now calculate the velocity $v(t)$ at all times from $t = t_1$ to $t = t_N$.

The acceleration at each time step is

$$a(t_k) = g - \left(\frac{\beta}{m}\right) v(t_k)$$

The displacement at each time step is

$$v = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t}$$
$$v(t_{k+1}) = \frac{x(t_{k+2}) - x(t_k)}{2\,\Delta t}$$

$$x(t_{k+2}) = x(t_k) + (2\,\Delta t)\,v(t_{k+1})$$

**EXAMPLE** $F_R = -\beta v \qquad m a = m g - \beta v$

The mscript **mec_fr_mg_bv.m** can be used for simulations for the motion of an object acted upon a resistive force of the form $F_R = -\beta v$ (Model 1).

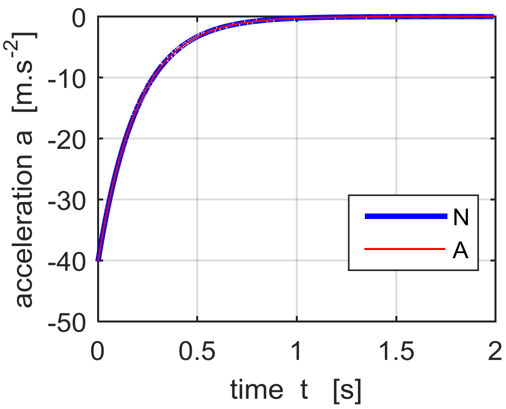
Input parameters

$m$ = 2 kg $\beta$ = 10 kg.s$^{-1}$ $v_0$ = 10 m.s$^{-1}$ $\Delta t$ = 1x10$^{-4}$ s $t_{max}$ = 2.00 s

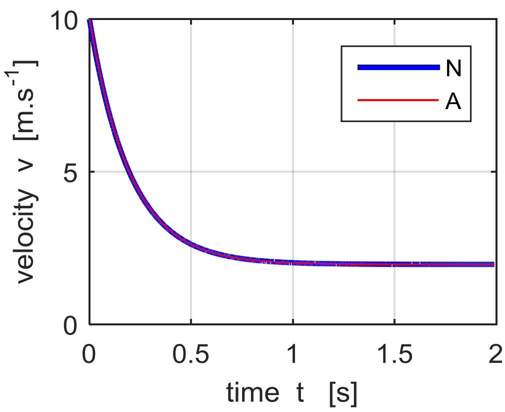
Outputs **N** numerical approach **A** analytical approach

terminal velocity $v_T$ = 1.96 m.s$^{-1}$

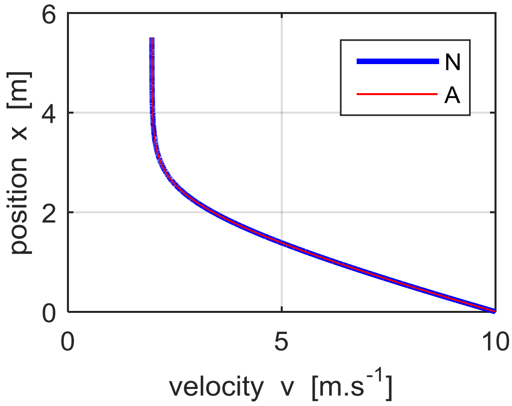
acceleration $t \to \infty \quad a \to 0$

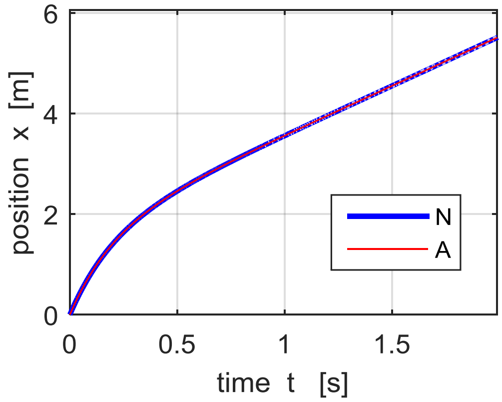



velocity $t \to \infty \quad v \to v_T$



displacement $t \to \infty \quad x \to \infty$

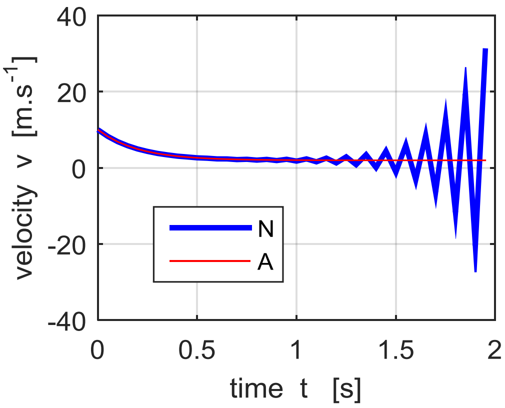



For the input parameters used in this simulation, there is excellent agreement between the values calculated using the numerical and analytical approaches.

However, you always need to be careful in using numerical approaches to solve problems. In this instance, you need to check the convergence of results by progressively making the time step $\Delta t$ smaller.

When the time step is $\Delta t$ = 5x10$^{-2}$ s the numerical and analytical results do not agree. The time step is too large for accurate results using the numerical approach.



## MODEL 2 $F_R = -\alpha v^2 \qquad m a = m g - \alpha v^2$

The mscript used for the plots is **mec_fr_mg_bv2.m**

### Analytical Approach $F_R = -\alpha v^2 \qquad m a = m g - \alpha v^2$

For the vertical motion of an object through a fluid, the forces acting on the object are the gravitational force $F_G$ (weight) and the resistive force $F_R$. In our frame of reference, down is the positive direction.

The equation of motion of the object is determined from Newton's Second Law.

$$m a = m \frac{dv}{dt} = F_G - F_R = mg - \alpha v^2 \left( v / |v| \right) \qquad a = g - \frac{\alpha}{m} v^2 \left( v / |v| \right)$$

where $a$ is the acceleration of the object at any instance.

The initial conditions are $t = 0 \quad v = v_0 \quad x = 0 \quad a = g - (\alpha / m) v_0^{\,2} \left( \frac{v_0}{|v_0|} \right)$

When $a = 0$, the velocity is constant $v = v_T$ where $v_T$ is the **terminal velocity**

$$0 = m g - \alpha v_T^{\,2} \qquad v_T^{\,2} = \frac{m g}{\alpha}$$

$$v_T = \sqrt{\frac{m g}{\alpha}}$$

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions ($t = 0$ and $v = v_0$) and final conditions ($t$ and $v$).

Since the acceleration depends upon $v^2$ its a more difficult problem then for the linear resistive force example. We have to do separate analytical calculations for the motion when the object is falling or rising.

**Velocity of the object is always positive (falling object)** $v_0 \geq 0 \quad v \geq 0$

Equation of motion

$$a = g - \frac{\alpha}{m} v^2$$

$$a = \frac{dv}{dt} = g - \left(\frac{\alpha}{m}\right) v^2$$

$$dt = \frac{dv}{g - \left(\frac{\alpha}{m}\right) v^2} = \frac{dv}{\left(\frac{\alpha}{m}\right)\left(\left(\frac{m\,g}{\alpha}\right) - v^2\right)} \qquad {v_T}^2 = \frac{m\,g}{\alpha}$$

$$-\left(\frac{\alpha}{m}\right) dt = \frac{dv}{v^2 - {v_T}^2} = \left(\frac{1}{2v_T}\right)\left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\left(2\sqrt{\frac{m\,g}{\alpha}}\right)\left(\frac{\alpha}{m}\right) dt = \left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}}\; dt = \left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}} \int_0^t dt = \int_{v_0}^{v} \left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}}\, t = \left[-\log_e(v - v_T) - \log_e(v + v_T)\right]_{v_0}^{v}$$

$$\sqrt{\frac{4\,\alpha\,g}{m}}\, t = \left[\log_e(v - v_T) + \log_e(v + v_T)\right]_{v_0}^{v}$$

$$\sqrt{\frac{4\,\alpha\,g}{m}}\, t = \left[\log_e\left(\frac{v - v_T}{v_0 - v_T}\right) + \log_e\left(\frac{v - v_T}{v_0 - v_T}\right)\right]$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}}\, t = \log_e\left\{\left(\frac{v - v_T}{v_0 - v_T}\right)\left(\frac{v_0 + v_T}{v + v_T}\right)\right\}$$

$$\left(\frac{v - v_T}{v_0 - v_T}\right)\left(\frac{v_0 + v_T}{v + v_T}\right) = e^{-\sqrt{\frac{4\alpha g}{m}}\,t} \qquad \sqrt{\frac{4\,\alpha\,g}{m}} = \sqrt{\frac{4\,\alpha\,g^2}{mg}} = \sqrt{\frac{4\,g^2}{{v_T}^2}} = \frac{2\,g}{v_T}$$

$$v - v_T = (v + v_T)\left(\frac{v_0 - v_T}{v_0 + v_T}\right) e^{-\frac{2g}{v_T}t} = (v + v_T)K \qquad K = \left(\frac{v_0 - v_T}{v_0 + v_T}\right) e^{-\frac{2g}{v_T}t}$$

$$v(1 - K) = v_T\,(1 + K) \qquad v = v_T\left(\frac{1 + K}{1 - K}\right)$$

$$v = v_T\left(\frac{1 + \left(\frac{v_0 - v_T}{v_0 + v_T}\right) e^{-\frac{2g}{v_T}t}}{1 - \left(\frac{v_0 - v_T}{v_0 + v_T}\right) e^{-\frac{2g}{v_T}t}}\right)$$

$$v = v_T\left(\frac{(v_0 + v_T) + (v_0 - v_T)\,e^{-\frac{2g}{v_T}t}}{(v_0 + v_T) - (v_0 - v_T)\,e^{-\frac{2g}{v_T}t}}\right)$$

valid only if $v_0 \ge 0 \quad v \ge 0$

We can now calculate the displacement $x$ as a function of velocity $v$

$$a = \frac{dv}{dt} = \frac{v\,dv}{dx} = g - (\alpha / m)v^2$$

$$\frac{v\,dv}{dx} = (\alpha / m)(m\,g/\alpha - v^2) \qquad {v_T}^2 = m\,g/\alpha$$

$$dx = \left(\frac{m}{\alpha}\right)\frac{v\,dv}{({v_T}^2 - v^2)} = \left(\frac{-{V_T}^2}{2\,g}\right)\frac{(-2v)\,dv}{({v_T}^2 - v^2)}$$

$$\int_0^x dx = \left(\frac{-{v_T}^2}{2\,g}\right)\int_{v_0}^{v}\frac{(-2v)}{({v_T}^2 - v^2)}dv$$

$$x = \left(\frac{-{v_T}^2}{2\,g}\right)\left[\log_e({v_T}^2 - v^2)\right]_{v_0}^{v}$$

$$x = \left(\frac{{v_T}^2}{2\,g}\right)\log_e\left(\frac{{v_T}^2 - {v_0}^2}{{v_T}^2 - v^2}\right)$$

valid only if $v_0 \geq 0 \quad v \geq 0$

We can now investigate what happens as time $t \to \infty$

$$v(t \to \infty) = v_T\left(\frac{(v_0 + v_T) + 0}{(v_0 + v_T) - 0}\right) \qquad e^{-\frac{2g}{v_T}t} \to 0$$

$$v(t \to \infty) = v_T$$

In falling, the object will finally reach a constant velocity $v_T$ ($a = 0$ ) which is known as the terminal velocity.

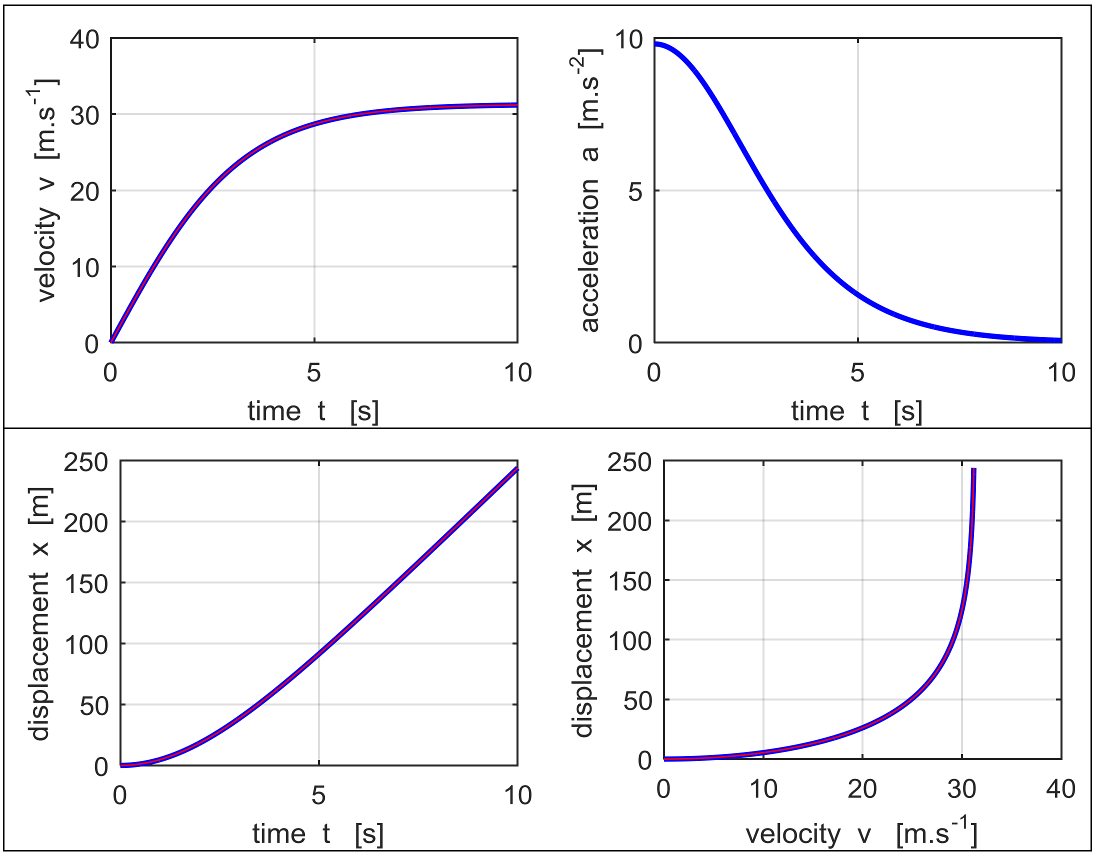
$$t \to \infty \quad v \to v_T \quad v_T - v \to 0 \quad \frac{1}{v_T - v} \to \infty$$

$$x = \left(\frac{{V_T}^2}{2\,g}\right)\log_e\left(\frac{{v_T}^2 - {v_0}^2}{{v_T}^2 - v^2}\right) \to \infty$$

In falling, as time $t$ increases the objects displacement $x$ just gets larger and larger.

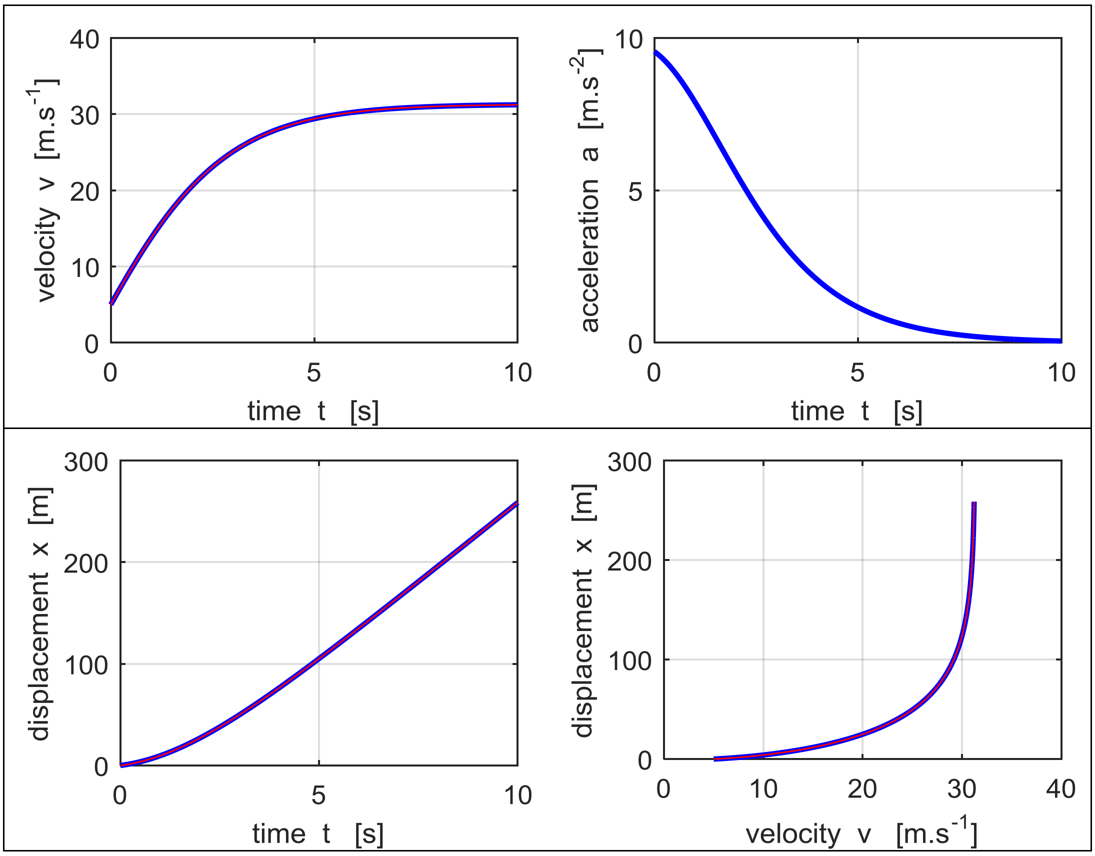
**Example** Small rock dropped from rest:

$$m = 0.010\ \text{kg} \quad \alpha = 1.00\times10^{-4}\ \text{kg.m}^{-1} \quad v_0 = 0\ \text{m.s}^{-1} \quad \Rightarrow \quad v_T = 31.3\ \text{m.s}^{-1}$$



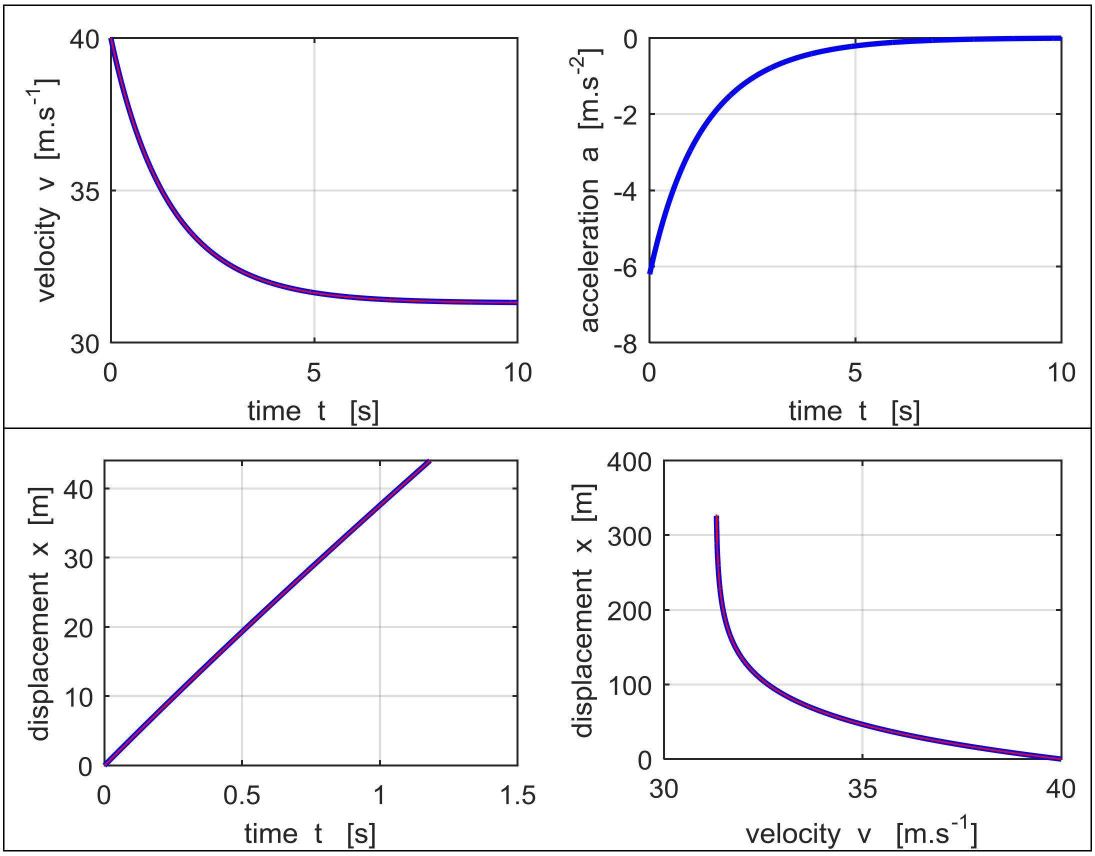
**Example** Small rock thrown vertically downward ($v < v_T$)

$$m = 0.010 \text{ kg} \quad \alpha = 1.00\times10^{-4} \text{ kg.m}^{-1} \quad v_0 = +5.00 \text{ m.s}^{-1} \quad \Rightarrow \quad v_T = 31.3 \text{ m.s}^{-1}$$



**Example** Small rock thrown vertically downward ($v > v_T$)

$$m = 0.010 \text{ kg} \quad \alpha = 1.00\times10^{-4} \text{ kg.m}^{-1} \quad v_0 = +40.0 \text{ m.s}^{-1} \quad \Rightarrow \; v_T = 31.3 \text{ m.s}^{-1}$$




For problems in which the object is projected vertically upward, you have to divide the problem into two parts. (1) Calculate the time to reach its maximum height and calculate the maximum height reached for the upward motion. (2) Reset the initial conditions to the position at maximum height where the initial velocity becomes $v_0$ = 0 and do the calculations for the downward movement of the object.

## Velocity of the object negative and moving up $v_0 < 0 \quad and \quad v < 0$

Equation of motion Note: up is the positive direction

$$a = g + \frac{\alpha}{m} v^2 \qquad \text{valid only if} \quad v_0 < 0 \quad and \quad v < 0$$

$$a = \frac{dv}{dt} = g + \left(\frac{\alpha}{m}\right) v^2$$

$$dt = \frac{dv}{g + \left(\frac{\alpha}{m}\right) v^2} = \frac{dv}{\left(\frac{\alpha}{m}\right)\left(\left(\frac{m\,g}{\alpha}\right) + v^2\right)} \qquad {v_T}^2 = \frac{m\,g}{\alpha}$$

$$\left(\frac{\alpha}{m}\right) dt = \frac{dv}{v^2 + {v_T}^2}$$

$$\left(\frac{\alpha}{m}\right) \int_0^t dt = \int_{v_0}^{v} \frac{dv}{v^2 + {v_T}^2}$$

$$\text{Standard Integral} \qquad \int \frac{dx}{a^2 + x^2} = \left(\frac{1}{a}\right) \text{atan}\left(\frac{x}{a}\right) + C$$

$$\left(\frac{\alpha}{m}\right) t = \left(\frac{1}{v_T}\right) \left[\text{atan}\left(\frac{v}{v_T}\right)\right]_{v_0}^{v} \qquad \left(\frac{m}{\alpha\, v_T}\right) = \left(\frac{m}{\alpha\, v_T}\frac{g}{g}\right) = \left(\frac{v_T}{g}\right)$$

$$t = \left(\frac{v_T}{g}\right) \left[\text{atan}\left(\frac{v}{v_T}\right)\right]_{v_0}^{v} = \left(\frac{v_T}{g}\right) \left[\text{atan}\left(\frac{v}{v_T}\right) - \text{atan}\left(\frac{v_0}{v_T}\right)\right]$$

The time tup to reach maximum height occurs when $v = 0$

$$t_{up} = \left(\frac{v_T}{g}\right) \left[\text{atan}\left(\frac{0}{v_T}\right) - \text{atan}\left(\frac{v_0}{v_T}\right)\right] = -\left(\frac{v_T}{g}\right) \text{atan}\left(\frac{v_0}{v_T}\right) \qquad v_0 < 0$$

The velocity $v$ as a function of time $t$ is

$$\text{atan}\left(\frac{v}{v_T}\right) = \text{atan}\left(\frac{v_0}{v_T}\right) + \left(\frac{g}{v_T}\right) t$$

$$v = v_T \tan\left[\text{atan}\left(\frac{v_0}{v_T}\right) + \left(\frac{g}{v_T}\right) t\right] \qquad \text{atan}\,\theta \equiv \tan^{-1}\theta \qquad v_0 < 0 \quad and \quad v < 0$$

The displacement $x$ as a function of velocity $v$ is

$$a = \frac{dv}{dt} = \frac{v\,dv}{dx} = g + (\alpha / m)v^2$$

$$\frac{v\,dv}{dx} = (\alpha / m)\left(m\,g/\alpha - v^2\right) \qquad {v_T}^2 = m\,g/\alpha$$

$$dx = \left(\frac{m}{\alpha}\right)\frac{v\,dv}{\left({v_T}^2 + v^2\right)} = \left(\frac{{v_T}^2}{2\,g}\right)\frac{(2v)\,dv}{\left({v_T}^2 + v^2\right)}$$

$$\int_0^x dx = \left(\frac{{v_T}^2}{2\,g}\right)\int_{v_0}^{v} \frac{(2v)}{\left({v_T}^2 + v^2\right)}dv \qquad\qquad v_0 < 0 \quad \mathit{and} \quad v < 0$$

$$x = \left(\frac{{v_T}^2}{2\,g}\right)\left[\log_e\left({v_T}^2 + v^2\right)\right]_{v_0}^{v}$$

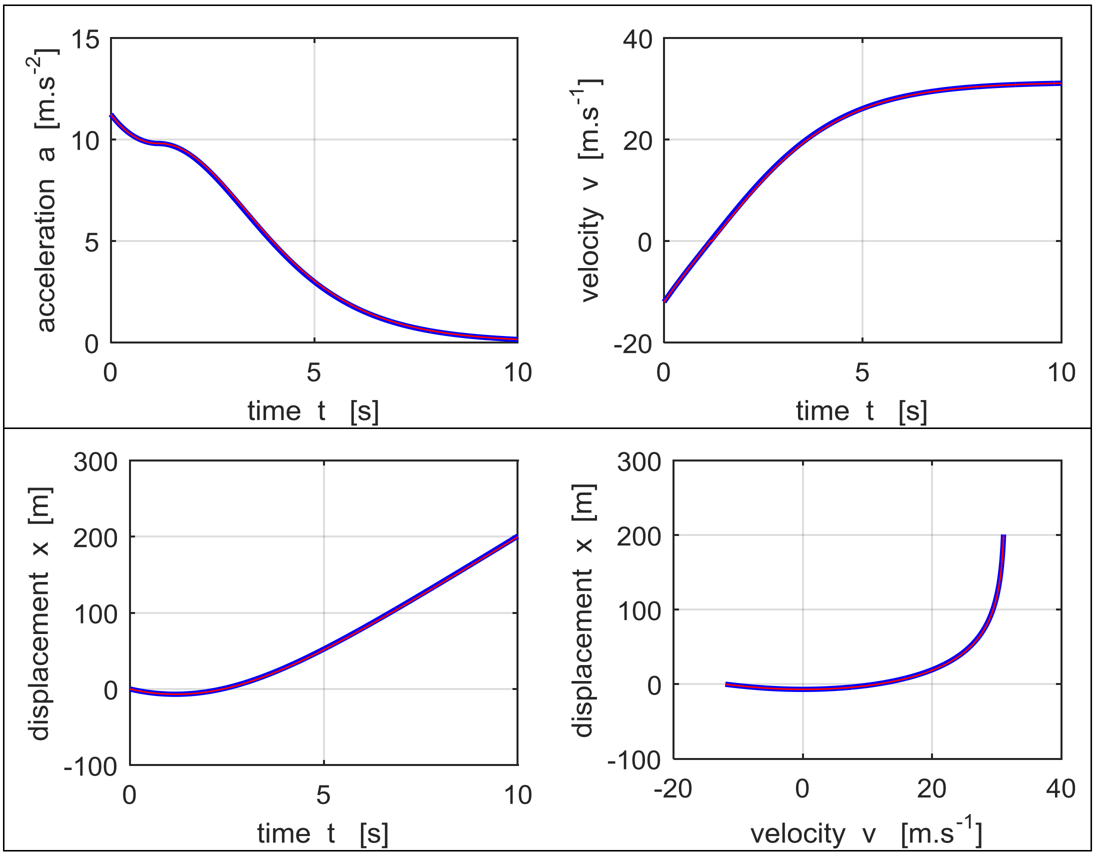
$$x = \left(\frac{{v_T}^2}{2\,g}\right)\log_e\left(\frac{{v_T}^2 + v^2}{{v_T}^2 + {v_0}^2}\right)$$

The maximum height $x_{up}$ reached by the object occurs when $v = 0$

$$x_{up} = \left(\frac{{v_T}^2}{2\,g}\right)\log_e\left(\frac{{v_T}^2}{{v_T}^2 + {v_0}^2}\right)$$

**Example** Small rock thrown vertically upward ($v_0 < 0$ $v_0 = -u$ $u > 0$)

$m = 0.010$ kg $\alpha = 1.00 \times 10^{-4}$ kg.m$^{-1}$ $v_0 = -12.0$ m.s$^{-1}$ $\Rightarrow$ $v_T = 31.3$ m.s$^{-1}$




The terminal velocity $v_T$ is

$$v_T^{\,2} = m\,g/\alpha$$

$$v_T = \sqrt{m\,g/\alpha} = \sqrt{\left(10^{-2}\right)\left(9.8\right)/\left(10^{-4}\right)}\ \text{m.s}^{-1}$$

$$v_T = 31.31\ \text{m.s}^{-1}$$

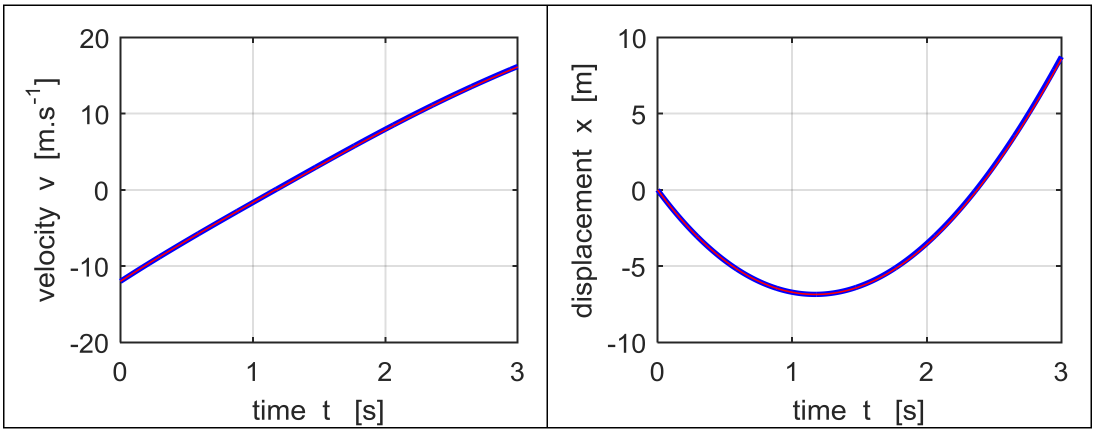
When $v = 0$ the object reaches its maximum height $x_{up}$ (up is negative)

$$x_{up} = \left(\frac{v_T^{\,2}}{2\,g}\right)\log_e\left(\frac{v_T^{\,2}}{v_T^{\,2}+v_0^{\,2}}\right)$$
$$x_{up} = -6.855\ \mathrm{m}$$

The time $t_{up}$ to reach the maximum height

$$t_{up} = -\left(\frac{v_T}{g}\right)\mathrm{atan}\left(\frac{v_0}{v_T}\right)$$
$$t_{up} = 1.169\ \mathrm{s}$$

The calculations agree with the values for $t_{up}$ and $x_{up}$ determined from the graphs.




From the graphs:

$x = 0$ time $t = 2.275$ s velocity $v = 11.21$ $\mathrm{m.s^{-1}}$

Time to fall from max height to origin $t_{down} = (2.375 - 1.169)\ \mathrm{s} = 1.206\ \mathrm{s}$

takes slight longer to fall then rise to and from origin to max height

Launch speed = 12.00 $\mathrm{m.s^{-1}}$ slightly greater than return speed = 11.21 $\mathrm{m.s^{-1}}$

In the absence of any resistive forces $a = g \quad v = v_0 + a\,t \quad v^2 = v_0^{\,2} + 2\,a\,s$

At maximum height $v = 0 \quad t_{up} = 1.2245\ \mathrm{s} \quad x_{up} = -7.3469\ \mathrm{m}$

## Numerical Approach $F_R = -\alpha v^2$ $ma = mg - \alpha v^2$

We can also find the velocity and displacement of the object by solving Newton's Second Law of motion using a finite difference method.

We start with

(4) $$a = \frac{dv}{dt} = g - (\alpha / m) v^2 \left( \frac{v}{|v|} \right)$$

In the finite difference method we calculate the velocity $v$ and displacement $x$ at $N$ discrete times $t_k$ at fixed time intervals $\Delta t$

$t_1, t_2, \ldots, t_k, \ldots, t_N$ $\Delta t = t_2 - t_1$ $t_1 = 0$ $t_k = (k\text{-}1)\,\Delta t$ $k = 1, 2, 3, \ldots, N$

The acceleration $a$ is approximated by the difference formula

$$\frac{dv(t_{k+1})}{dt} \approx \frac{v(t_{k+2}) - v(t_k)}{2\,\Delta t}$$

Therefore, the velocity $v(t_{k+2})$ at time $t_k$+2 is

$$\frac{v(t_{k+2}) - v(t_k)}{2\,\Delta t} = g - (\alpha / m) v(t_{k+1})^3 / |v|$$

$$v(t_{k+2}) = v(t_k) + (2\,\Delta t)(\,g - \alpha / m) v(t_{k+1})^3 / |v|$$

$$a(t_k) = \frac{dv}{dt} = g - (\alpha / m) v(t_k)^3 / |v(t_k)|$$

Hence, to calculate the velocity $v(t_{k+2})$ we need to know the velocity at the two previous time steps $t_{k+1}$ and $t_k$. We know $t_1 = 0$ and $v(t_1) = v(0) = v_0$.

We estimate the velocity at the second time step $t_2$

(13) $$v(t_2) = v(t_1) + a(t_1)\Delta t$$

where we have assumed a constant acceleration in the first time step. We can improve our estimate of $v(t_2)$ by using an average value of the acceleration in the first time step

$$v(t_1) + \left[ \frac{a(t_1) + a(t_2)}{2} \right] \Delta t \to v(t_2)$$

We can now calculate the velocity $v(t)$ at all times from $t = t_1$ to $t = t_N$.

The displacement at each time step is

$$v = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t}$$

$$v(t_{k+1}) = \frac{x(t_{k+2}) - x(t_k)}{2\,\Delta t}$$

(15) $$x(t_{k+2}) = x(t_k) + (2\,\Delta t)\,v(t_{k+1})$$

Provided the time step is small enough, there is excellent agreement between the numerical values and analytical values for the acceleration, velocity and displacement.

# REYNOLDS NUMBER and MOTION THROUGH A FLUID

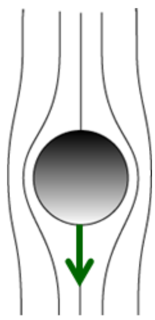
To characterize the motion of an object moving through a fluid, it is useful to define a dimensionless quantity called the **Reynolds number** $N_R$

$$N_R = \left(\frac{2\,\rho_F\,R}{\eta_F}\right) v$$

where the fluid is characterized by its viscosity $\eta_F$ and density $\rho_F$ . The effective radius of the object is $R$ and its velocity is $v$. Different values of the Reynolds number $N_R$ determine the different regimes of flow in which different laws of the resistive force are valid.

| **Reynolds number $N_R$** | **Resistive force** |
|---|---|
| 0 to 10 | Stokes' law $F_{drag} = (6\,\pi\,\eta_F\,R)\,v$ |
| 10 to 300 | Transition region |
| 300 to 300 000 | constant drag coefficient $C_D$<br>$F_{drag} \sim \left(\frac{1}{2} C_D\,A\,\rho_F\right) v^2$ |
| > 300 000 | Some variation in drag coefficient $C_D$ with $v$<br>$F_{drag}$ only roughly proportional to $v_2$ |

## Low Reynolds Number (0 < $N_R$ < 10)



For a small object moving through a fluid with a low velocity, the flow around the object is essentially **laminar** where the fluid flows in layers and there is no turbulence. In this situation, it is found experimentally that the **viscous drag force** $F_{drag}$ acting on the object by the fluid is directly proportional to the velocity $v$ of the object. For a small sphere of radius $R$, the drag force $F_{drag}$ is also directly proportional to the radius $R$.

For a small sphere of radius $R$, the viscous drag force $F_{drag}$ is given by **Stokes' Law**

viscous drag force $\quad F_{drag} = (6\,\pi\,\eta_F\,R)\,v \quad$ **STOKES LAW**

where the drag force $F_{drag}$ is always in the opposite direction to the velocity $v$.

The quantity $\eta_F$ is called the **viscosity** of the fluid [Pa.s kg.m$^{-1}$.s$^{-1}$]. Viscosity is a measure of the frictional force acting on a fluid flowing over a solid surface.

For an object falling through a fluid, we make the assumption that there are three distinct forces acting on the object

| | | |
|---|---|---|
| weight | $F_G = m\,g$ | |
| viscous drag force | $F_{drag} = -\left(6\,\pi\,\eta_F\,R\right)v$ | **STOKES' LAW** |
| buoyancy force | $F_{buoy} = -\left(\tfrac{4}{3}\pi\,R^3\right)\rho_F\,g$ | |

where in our frame of reference, down is taken as the positive direction.

Newton's Second Law applied to the falling object gives

$$m\,a = m\,g - \left(6\,\pi\,\eta_F\,R\right)v - \left(\tfrac{4}{3}\pi\,R^3\right)\rho_F\,g$$

The acceleration $a$ is

$$a = g - \left(6\,\pi\,\eta_F\,R/m\right)v - \left(\tfrac{4}{3}\pi\,R^3\right)\rho_F\,g/m$$

The terminal velocity $v_T$ ($a$ = 0) of the object is

$$v_T = \frac{m\,g - \left(\tfrac{4}{3}\pi\,R^3\right)\rho_F\,g}{6\,\pi\,\eta_F\,R}$$

The motion of tiny droplets of water falling through the air, tiny particles of rock settling under the sea and the sedimentation of red blood cells in blood plasma, the rise of oil drops in water are examples of the motion of objects moving through a fluid with low Reynolds numbers.

We can use the finite difference method to numerically find the velocity $v$ and displacement $x$ of the object as functions of time.

$$a\left(t_{k+1}\right) = \frac{dv\left(t_{k+1}\right)}{dt} \approx \frac{v\left(t_{k+2}\right) - v\left(t_k\right)}{2\,\Delta t}$$

$$\frac{v\left(t_{k+2}\right) - v\left(t_k\right)}{2\,\Delta t} = g - \left(6\,\pi\,\eta\,R/m\right)v\left(t_{k+1}\right) - \left(\tfrac{4}{3}\pi\,R^3\right)\rho\,g/m$$

$$v\left(t_{k+2}\right) = v\left(t_k\right) + \left(2\,\Delta t\right)\left(g - \left(6\,\pi\,\eta\,R/m\right)v\left(t_{k+1}\right) - \left(\tfrac{4}{3}\pi\,R^3\right)\rho\,g/m\right)$$

**Example: A small oil drop released in water will rise to the surface**

mscript **mec_stokes.m**

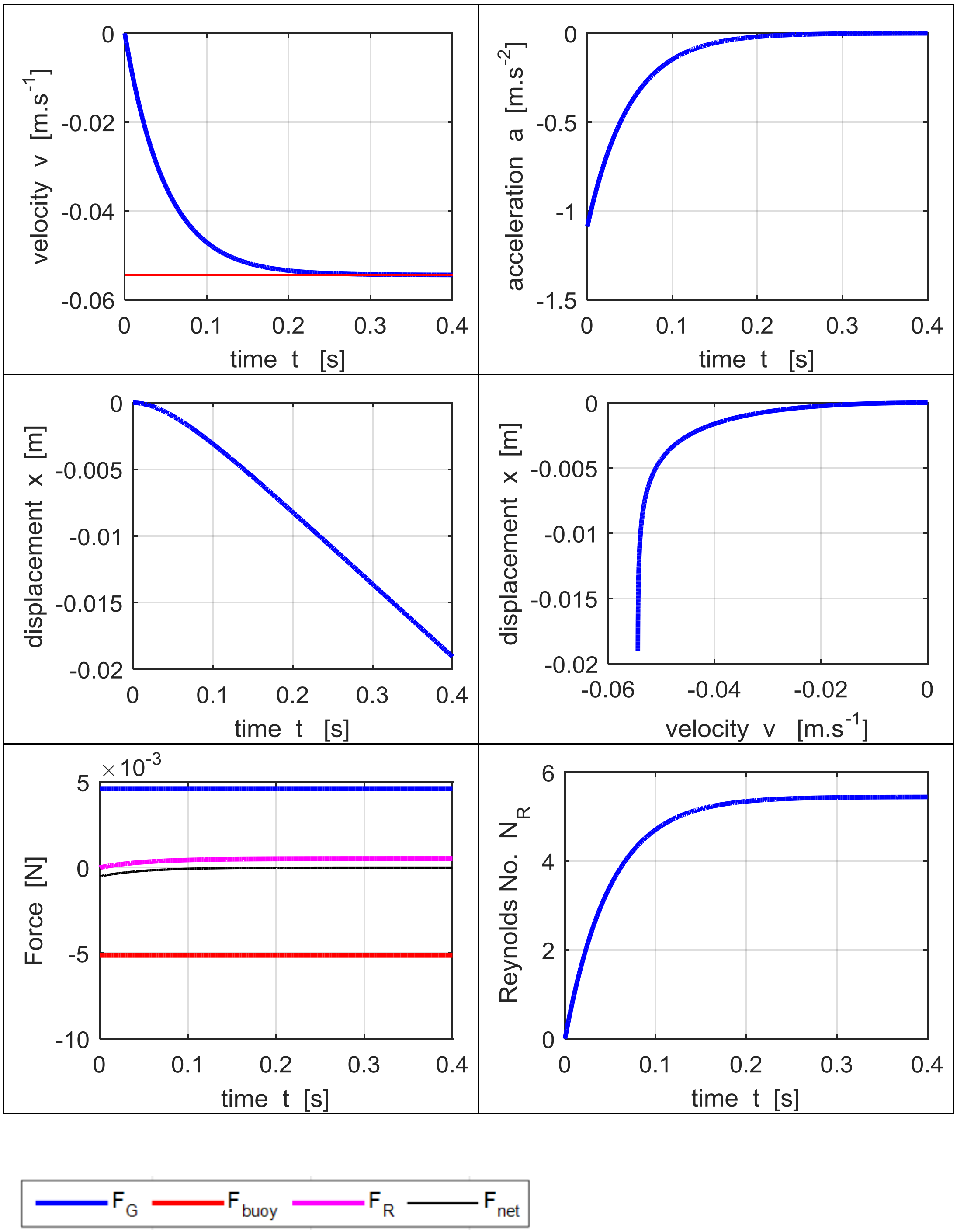
Input parameters

| | |
|---|---|
| radius of oil drop | $R$ = 5x10$^{-3}$ m |
| density of oil drop | $\rho$ = 900 kg.m$^{-3}$ |
| density of water | $\rho_F$ = 1000 kg.m$^{-3}$ |
| viscosity of water | $\eta_F$ = 0.1 Pa.s |
| max time | $t_{Max}$ = 0.4 s |
| No. time steps | $N$ = 10000 |

Output parameters

| | |
|---|---|
| mass of oil drop | $m$ = 4.71x10$^{-4}$ kg |
| terminal velocity | $\lvert v_T \rvert$ = 0.054 m.s$^{-1}$ |
| time to reach $v_T$ | $t(v_T)$ ~ 0.3 s |

Reynolds number is in the range from 0 to 10, hence Stokes' Law $\left(F_R \propto v\right)$ is valid. The main reason the oil drop rises is because of the buoyancy force, the viscous drag force plays only a minor role.


velocity v [m.s-1]
time t [s]
acceleration a [m.s-2]
time t [s]
displacement x [m]
time t [s]
displacement x [m]
velocity v [m.s-1]
×10-3
Force [N]
time t [s]
Reynolds No. NR
time t [s]
FG
Fbuoy
FR
Fnet

### Example: How do small water droplets fall?

mscript **mec_stokes.m**

Input parameters

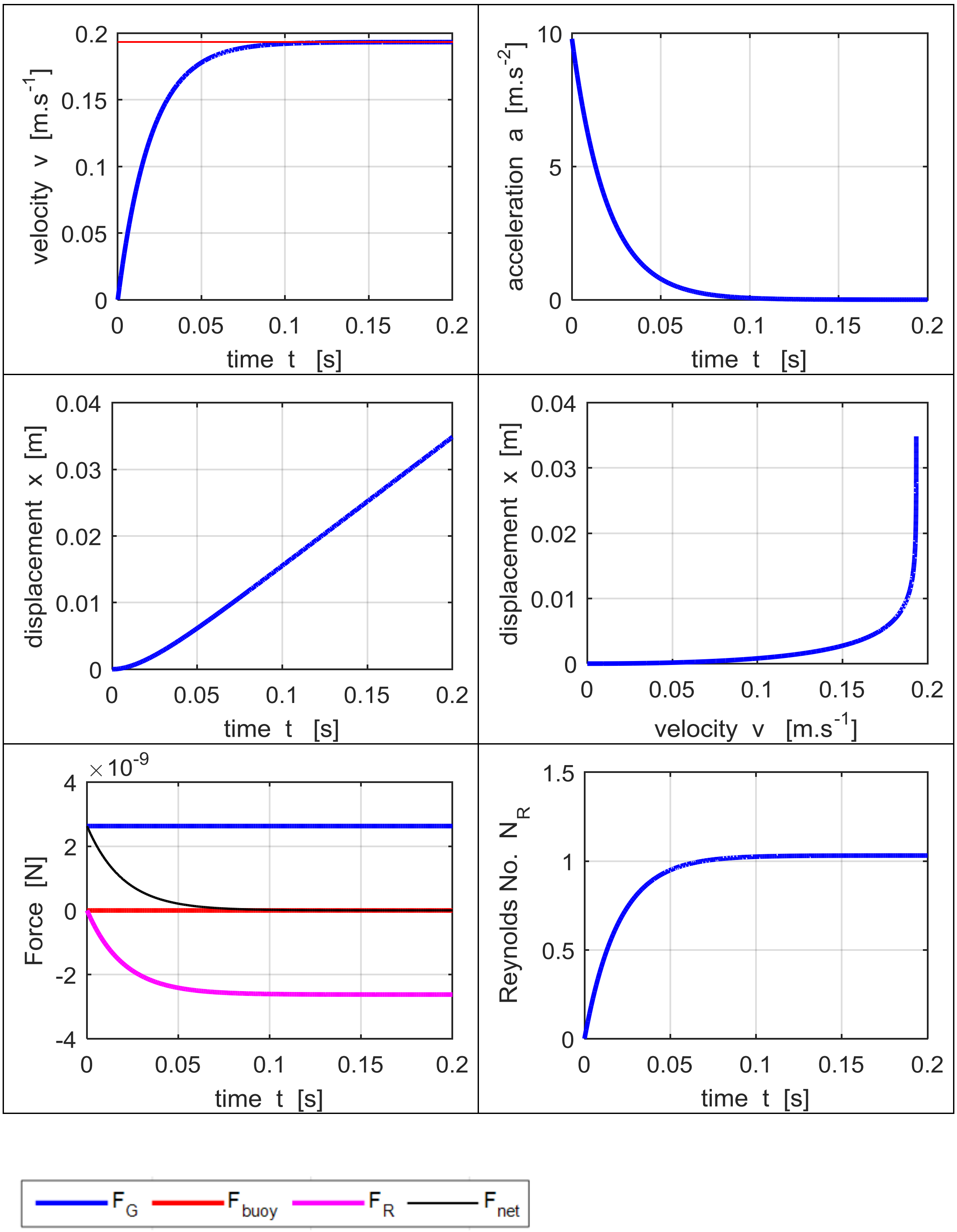
| | |
|---|---|
| radius of water drop | $R$ = 4x10$^{-5}$ m |
| density of water drop | $\rho$ = 1000 kg.m$^{-3}$ |
| density of air | $\rho_F$ = 1.20 kg.m$^{-3}$ |
| viscosity of air | $\eta_F$ = 1.80x10$^{-5}$ Pa.s |
| max time | $t_{Max}$ = 1.0 s |
| No. time steps | $N$ = 10000 |

Output parameters

| | |
|---|---|
| mass of water drop | $m$ = 2.7x10$^{-10}$ kg |
| terminal velocity | $\lvert v_T \rvert$ = 0.2 m.s$^{-1}$ |
| time to reach $v_T$ | $t(v_T)$ ~ 0.1 s |

Reynolds number is in the range from 0 to 1, hence Stokes' Law $(F_R \propto v)$ is valid.

In the upper atmosphere, water vapour condenses on hygroscopic nuclei (~ 10$^{10}$ / m$^3$ radii ~ 10$^{-7}$ m) forming water droplets whose radii range from about 4x10$^{-6}$ m to 4x10$^{-5}$ m. For $R <$ 4x10$^{-5}$ m, the water droplets very quickly each their very low values of terminal speed $v_T$ < 0.2 m.s$^{-1}$. Because of the very small values of the terminal speeds of the water droplets, they can be kept afloat by vertical air currents. The aggregation of suspended water droplets forms clouds or near ground level water droplets forms fogs.




In this example, the buoyance force acting on water droplets is insignificant.

Stokes' law is not valid for raindrops. A raindrop of only 1 mm in radius has a terminal velocity that is very large $v_T$ ~ 120 m.s$^{-1}$ and $N_R$ ~ 10 000.

## Intermediate Reynolds Number (300 < $N_R$ < 300 000)

For large fast moving objects such as cars, aeroplanes, cricket balls, baseballs, hailstones, and skydivers, the drag force to a good approximation varies with the square of its speed. The drag force $F_{drag}$ can be written as

$$F_{drag} = \left(\tfrac{1}{2} C_D\, A\, \rho_F\right) v^2$$

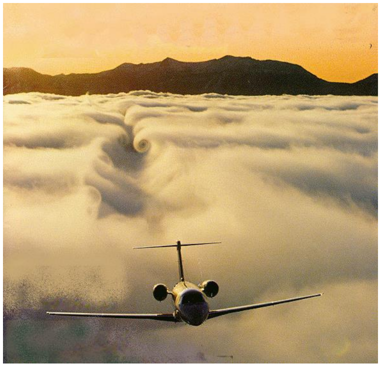
$A$ effective cross-sectional area of object [m$^2$]

$\rho_F$ density of fluid [kg.m$^{-3}$]

$C_D$ drag coefficient [dimensionless]

For objects of a define shape such as spheres, the drag coefficient is nearly constant over a wide range of speeds and sizes.

For larger Reynold numbers, the turbulence associated with the motion of the object through the fluid is mainly responsible for the frictional retarding force on the object as it moves through the fluid. This is the reason why the resistive force depends upon $v^2$ and it is much greater than the resistive force for laminar flow where the resistive force depends upon $v$.



For an object falling through a fluid, we make the assumption that there are two distinct forces acting on the object. The buoyant force can be assumed to be negligible.

| | |
|---|---|
| weight | $F_G = m\,g$ |
| drag force | $F_{drag} = \left(\tfrac{1}{2} C_D\, A\, \rho_F\right) v^2$ |

where in our frame of reference, down is taken as the positive direction.

Newton's Second Law applied to the falling object gives

$$m\,a = m\,g - \left(\tfrac{1}{2} C_D\, A\, \rho_F\right) v^2 \left(\frac{v}{|v|}\right)$$

The acceleration $a$ is

$$a = g - \left(\tfrac{1}{2} C_D\, A\, \rho_F / m\right) v^2 \left(\frac{v}{|v|}\right)$$

The terminal velocity $v_T$ ($a$ = 0) of the object is

$$v_T = \sqrt{\frac{2\,m\,g}{C_D\,A\rho_F}}$$

We can use the finite difference method to numerically find the velocity $v$ and displacement $x$ of the object as functions of time.

$$a\left(t_{k+1}\right) = \frac{dv\left(t_{k+1}\right)}{dt} \approx \frac{v\left(t_{k+2}\right) - v\left(t_k\right)}{2\,\Delta t}$$

$$\frac{v\left(t_{k+2}\right) - v\left(t_k\right)}{2\,\Delta t} = g - \left(\tfrac{1}{2}C_D\,A\,\rho_F\,/\,m\right)v^2\left(\frac{v\left(t_{k+1}\right)}{\left|v\left(t_{k+1}\right)\right|}\right)$$

$$v\left(t_{k+2}\right) = v\left(t_k\right) + \left(2\,\Delta t\right)\left[g - \left(\tfrac{1}{2}C_D\,A\,\rho_F\,/\,m\right)v^2\left(\frac{v\left(t_{k+1}\right)}{\left|v\left(t_{k+1}\right)\right|}\right)\right]$$

## Example Falling raindrops

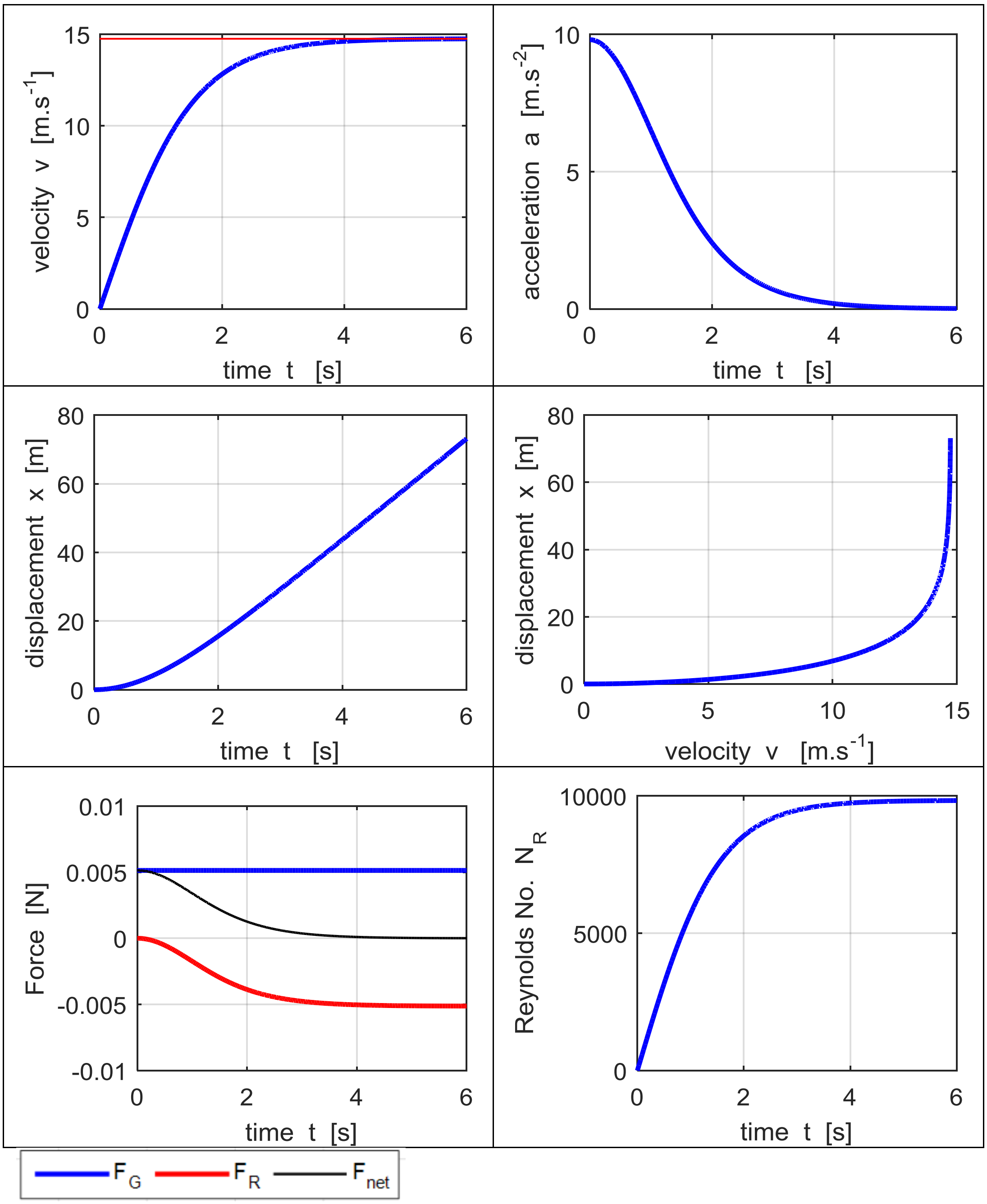
mscript **mec_drag.m**

Input parameters

| | |
|---|---|
| radius of raindrop | $R$ = 5.00x10$^{-3}$ m (big raindrop) |
| density of water drop | $\rho$ = 1000 kg.m$^{-3}$ |
| drag coefficient | $C_D$ = 0.500 |
| density of air | $\rho_F$ = 1.20 kg.m$^{-3}$ |
| viscosity of air | $\eta_F$ = 1.80x10$^{-5}$ Pa.s |
| max time | $t_{Max}$ = 6.00 s |
| No. time steps | $N$ = 10000 |

Output parameters

| | |
|---|---|
| mass of raindrop | $m$ = 5.24x10$^{-4}$ kg |
| cross-sectional area | $A$ = 7.85x10$^{-5}$ m$^2$ |
| terminal velocity | $\lvert v_T \rvert$ = 14.8 m.s$^{-1}$ |
| time to reach $v_T$ | $t(v_T)$ ~ 4 s |




Our model in assuming $F_{drag} \propto v^2$ should be OK since the Reynolds number varies from zero to about 10 000.

A raindrop could fall a distance ~ 5000 m. With no frictional forces acting the raindrops would reach the ground with speeds ~ 300 m.s$^{-1}$ (> 1000 km.s$^{-1}$) !!! This does not happen. Raindrops quickly each their terminal speed within seconds and reach the ground at speeds < 20 m.s$^{-1}$.

## Example A falling table tennis ball

mscript **mec_drag.m** **mec_tt_ball.m**

We can model a table tennis ball falling from rest using the numerical approach where the resistive force or drag force is

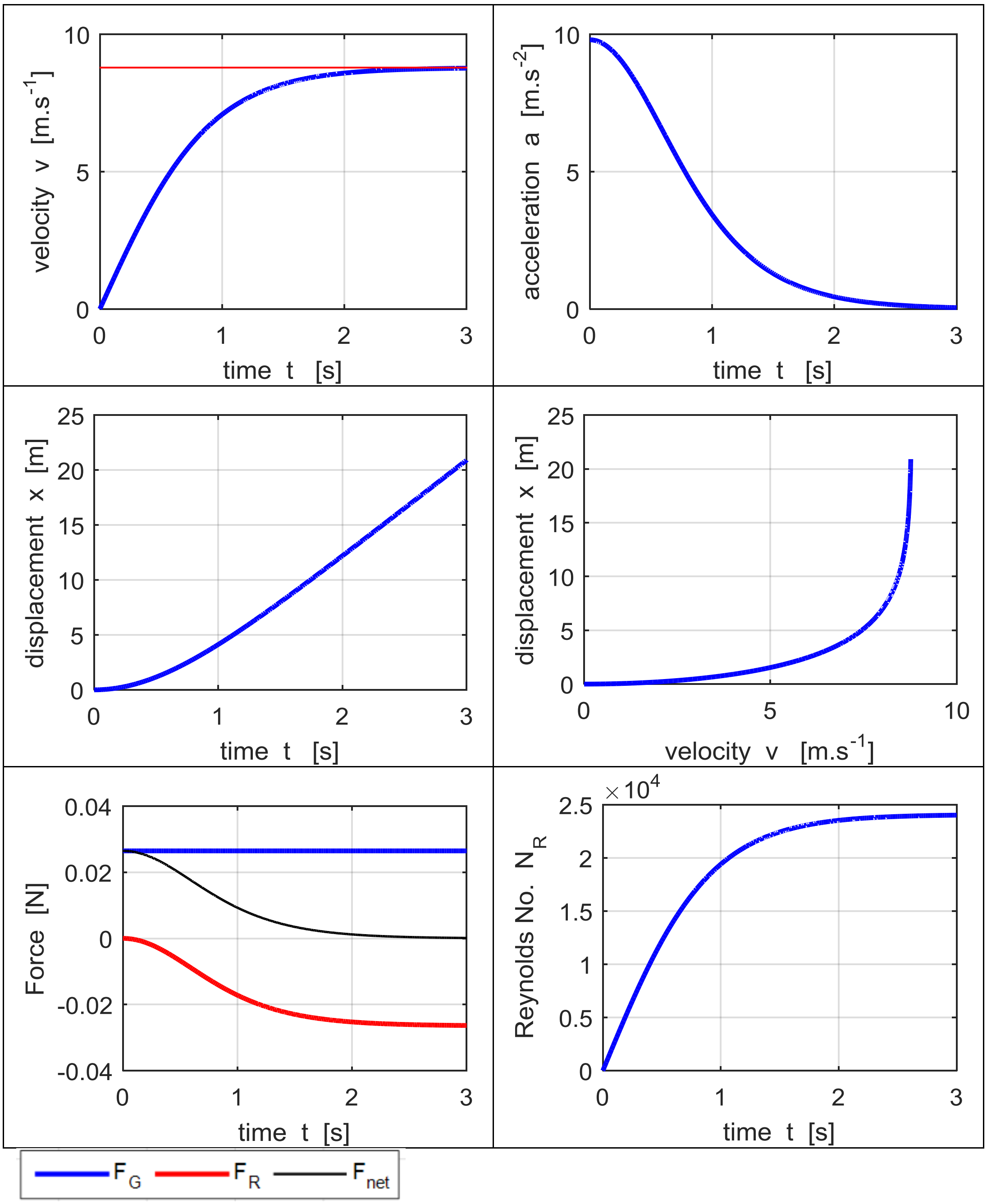
$$F_{drag} = \left(\tfrac{1}{2} C_D \, A \, \rho_F\right) v^2$$

### Input parameters

| | |
|---|---|
| radius of table tennis ball | $R$ = 20.0x10$^{-3}$ m |
| mass of table tennis ball | $m$ = 12.70x10$^{-3}$ kg<br>mass entered and not calculated from density |
| drag coefficient | $C_D$ = 0.445 |
| density of air | $\rho_F$ = 1.225 kg.m$^{-3}$ |
| viscosity of air | $\eta_F$ = 1.789x10$^{-5}$ Pa.s |
| max time | $t_{Max}$ = 3.00 s |
| No. time steps | $N$ = 10000 |

### Output parameters

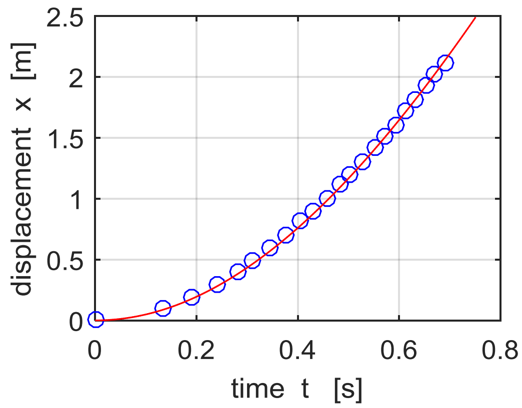
| | |
|---|---|
| cross-sectional area | $A$ = 1.26x10$^{-3}$ m$^2$ |
| terminal velocity | $\lvert v_T \rvert$ = 8.79 m.s$^{-1}$ |
| time to reach $v_T$ | $t(v_T)$ ~ 2.5 s |
| distance to reach $v_T$ | x($v_T$) ~ 17 m |




Our model in assuming $F_{drag} \propto v^2$ should be OK since the Reynolds number is mostly in the range from 300 to 300 000.

### *But, how good is our model?*

We can test our model against experimental data. The data for the fall of a table tennis ball was taken from a paper by French. The data for time and displacement of the falling table tennis ball was stored in the mscript **mec_tt_ball.m** and this mscript was used to compare the video measurements with our displacements / time predictions using the finite difference approximation to solve the equation of motion. The following plot shows the French data and the theoretical values for the displacement as functions of time.




The agreement between our model's predictions and the actual measured values is very good.

## Example A falling steel ball

mscript **mec_drag.m**

We can model a steel ball with a radius identical to that of a table tennis ball falling from rest using the numerical approach where the resistive force or drag force is

$$F_{drag} = \left(\tfrac{1}{2} C_D\, A\, \rho_F\right) v^2$$

We can then compare the fall of the steel ball with that of the table tennis ball.

### Input parameters

| | |
|---|---|
| radius of steel ball | $R$ = 20.0x10$^{-3}$ m |
| density of steel ball | $\rho$ = 7800 kg.m$^{-3}$ |
| mass of table tennis ball | $m$ = 12.70x10$^{-3}$ kg |
| drag coefficient | $C_D$ = 0.445 |
| density of air | $\rho_F$ = 1.225 kg.m$^{-3}$ |
| viscosity of air | $\eta_F$ = 1.789x10$^{-5}$ Pa.s |
| max time | $t_{Max}$ = 3.00 s |
| No. time steps | $N$ = 10000 |

| **Output parameters** | **table tennis ball** | **steel ball** | **free fall** $F_{drag}$ **= 0** |
|---|---|---|---|
| terminal velocity $v_T$ [m.s$^{-1}$] | 8.79 | 86 | $\infty$ |
| $t$ = 3.00 s $a$ [m.s$^{-2}$] | 0.049 | 8.75 | 9.80 |
| $t$ = 3.00 s $x$ [m] | 20.9 | 43.3 | 44.1 |
| $t$ = 3.00 s $v$ [m.s$^{-1}$] | 8.77 | 28.3 | 29.4 |
| $t$ = 3.00 s Reynolds number $N_R$ | 2.4x10$^4$ | 7.8x10$^4$ | |

It is "true" that heavier objects do fall faster than lighter ones.

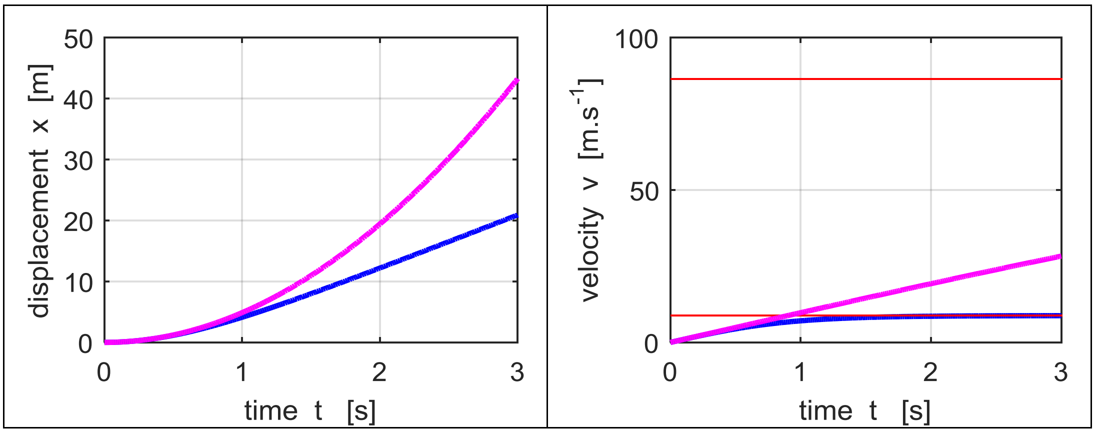



table tennis ball steel ball

## Example A falling skydiver



mscript **mec_drag.m**

We can model a falling skydiver falling from rest using the numerical approach where the resistive force or drag force is

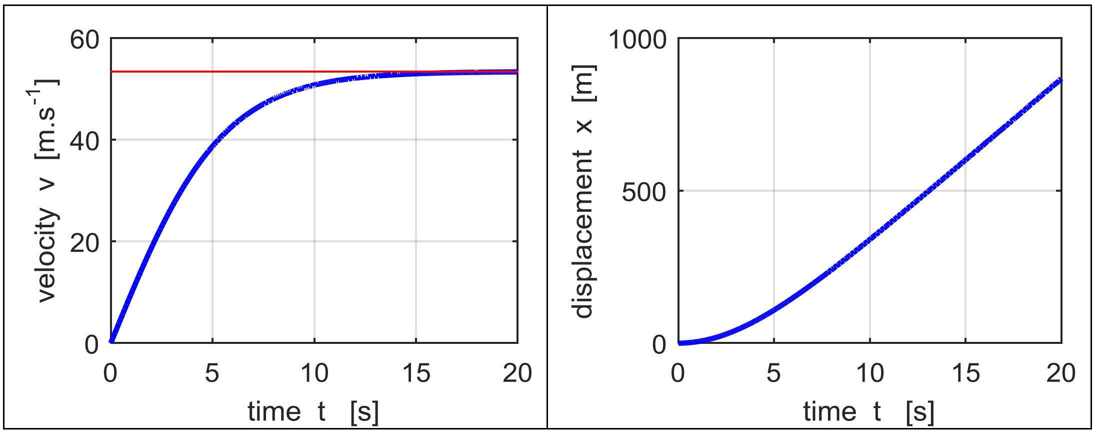
$$F_{drag} = \left(\tfrac{1}{2} C_D\, A\, \rho_F\right) v^2$$

### Input parameters

| | |
|---|---|
| radius (estimate) | $R$ = 0.500 m |
| mass | $m$ = 70 kg<br>mass entered and not calculated from density |
| drag coefficient | $C_D$ = 0.5 |
| density of air | $\rho_F$ = 1.225 kg.m$^{-3}$ |
| viscosity of air | $\eta_F$ = 1.789x10$^{-5}$ Pa.s |
| max time | $t_{Max}$ = 20.00 s |
| No. time steps | $N$ = 10000 |

### Output parameters

| | |
|---|---|
| cross-sectional area | $A$ = 0.7854 m$^2$ |
| terminal velocity | $\lvert v_T \rvert$ = 54 m.s$^{-1}$ = 192 km.h$^{-1}$ |
| time to reach $v_T$ | $t(v_T)$ ~ 18 s |
| distance to reach $v_T$ | x($v_T$) ~ 760 m |




You can investigate how the skydiver can control their speed of fall by varying their effective cross-sectional area.