

DOING PHYSICS WITH MATLAB

RAYLEIGH-SOMMERFELD DIFFRACTION RECTANGULAR APERTURES

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[Matlab Download Directory](#)

op_rs_rxy_01.m

Calculation of the irradiance in a plane perpendicular to the optical axis for a uniformly illuminated rectangular aperture.

simpsonxy_coeff.m

Function to calculate Simpson's two-dimensional coefficients

fn_distancePQ.m

Function to calculate distance between two points

turningPoints.m

Function to find the zero crossings of a function and its maxima and minima

op_rs_rxy_02.m

Updated version of **op_rs_rxy_01.m**. Calculation of the irradiance in a plane perpendicular to the optical axis for a rectangular aperture. The irradiance is not scaled but calculated in W.m^{-2} . The energy per second from the aperture to the observation screen is calculated and the energy per second within a circle of radius equal to the position of the first minima in the X direction in the observation screen is also calculated.

simpson2d.m

Function to calculate the value of a two-dimensional integral using the Simpson's [2D] method.

fn_distancePQ.m

Function to calculate distance between two points

FRAUNHOFER DIFFRACTION – RECTANGULAR APERTURE

We will consider the Fraunhofer diffraction patterns for a single rectangular aperture that is uniformly illuminated. The X and Y widths the rectangular aperture are a_x and a_y respectively. The irradiance distribution I is given by the product of two single slit functions

$$(1) \quad I = I_o \left(\frac{\sin(v_{Px})}{v_{Px}} \right)^2 \left(\frac{\sin(v_{Py})}{v_{Py}} \right)^2 \quad \text{Fraunhofer diffraction}$$

where I_o is a normalizing constant and the optical coordinates v_{Px} and v_{Py} are

$$(2) \quad v_{Px} = \frac{1}{2} k a_x \sin \theta_x = \left(\frac{\pi}{\lambda} \right) a_x \sin \theta_x \quad v_{Py} = \frac{1}{2} k a_y \sin \theta_y = \left(\frac{\pi}{\lambda} \right) a_y \sin \theta_y$$

and the angles θ_x and θ_y define the direction of the diffracted ray

$$(3) \quad \sin \theta_x = \frac{x_p}{\sqrt{x_p^2 + z_p^2}} \quad \sin \theta_y = \frac{y_p}{\sqrt{y_p^2 + z_p^2}}$$

The geometry for the diffraction pattern from a rectangular aperture is shown in figure (1).

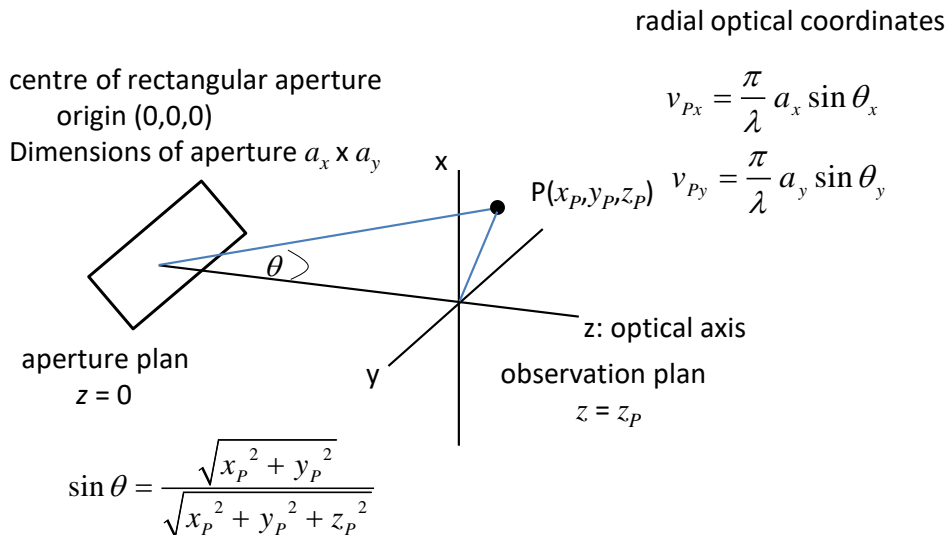


Fig. 1. Geometry for a rectangular aperture with dimensions $a_x \times a_y$. The radial coordinates are scaled perpendicular distance from the optical axis.

The resulting diffraction pattern for the irradiance has lines of zeros when

$$(3a) \quad v_{p_x} = m_x \pi \quad m_x = \pm 1, \pm 2, \pm 3, \dots \quad \text{and} \quad v_{p_y} = m_y \pi \quad m_y = \pm 1, \pm 2, \pm 3, \dots$$

or

$$(3b) \quad m_x \lambda = a_x \sin \theta_x \quad m_y \lambda = a_y \sin \theta_y$$

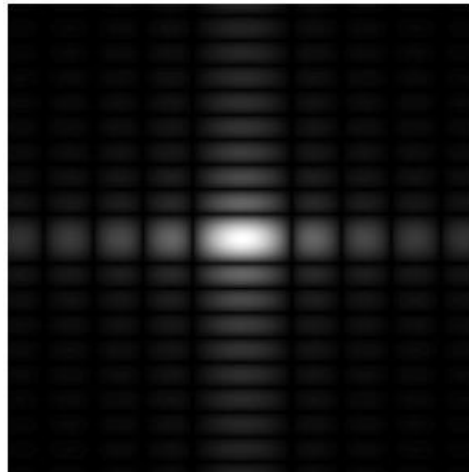


Fig. 1. Photograph like image for the Fraunhofer diffraction pattern of a rectangular aperture $a_y = 2 a_x$.

However, a more versatile approach to study the diffraction from an aperture is to evaluate the **Rayleigh-Sommerfeld diffraction integral of the first kind**. No approximations need to be made and the irradiance can be calculated in the near and far fields in an XY plane and along the Z axis right. There is excellent agreement between the far field predictions of the Fraunhofer theory and the results of evaluating the Rayleigh-Sommerfeld diffraction integral.

RAYLEIGH DIFFRACTION INTEGRAL OF THE FIRST KIND

The **Rayleigh-Sommerfeld region** includes the entire space to the right of the aperture. It is assumed that the Rayleigh-Sommerfeld diffraction integral of the first kind is valid throughout this space, right down to the aperture. There are no limitations on the maximum size of either the aperture or observation region, relative to the observation distance, because **no approximations have been made**.

The Rayleigh-Sommerfeld diffraction integral of the first kind (RS1) can be expressed as

$$(4) \quad E_P = \frac{1}{2\pi} \iint_{S_A} E_Q \frac{e^{jk r_{PQ}}}{r_{PQ}^3} z_p (jk r_{PQ} - 1) dS$$

where E_P is the electric field at the observation point P, E_Q is the electric field within the aperture and r_{PQ} is the distance from an aperture point Q to the point P. The double integral is over the area of the aperture S_A .

Approach 1 to evaluating The Diffraction Integral

op_rs_rxy_01.m

The double integral can be estimate numerically by a [two-dimensional form of Simpson's 1/3 rule](#). The electric field E_P at the point P is computed by

$$(5) \quad E_P = E_0 \sum_{m=1}^N \sum_{n=1}^N \left(S_{mn} E_{Qmn} \frac{e^{jk r_{PQmn}}}{r_{PQmn}^3} z_{Pmn} (jk r_{PQmn} - 1) \right)$$

where S_{mn} are the Simpson's two-dimensional coefficients and E_0 is a normalizing constant. Each term in equation (2) can be expressed as a matrix of size $N \times N$ and the matrices can be manipulated very easily in Matlab to give the estimate of the integral. The irradiance is proportional to the square of the magnitude of the electric field, hence the irradiance in the space beyond the aperture can be calculated by

$$(6) \quad I = I_0 |E^* E|$$

where I_0 is a normalizing constant and E^* is the complex conjugate of E .

Approach 2 to evaluating The Diffraction Integral

op_rs_rxy_02.m

Instead of using arbitrary units for electric field and irradiance we can perform all calculations in S.I. units.

The **irradiance** or more generally the term **intensity** has S.I. units of $\text{W}\cdot\text{m}^{-2}$. Another way of thinking about the irradiance is to use the term **energy density** as an alternative. The use of the letter I can be misleading, therefore, we will use the symbol u to represent the irradiance or energy density.

The irradiance or energy density u of a monochromatic light wave in matter is given in terms of its electric field E by

$$(7) \quad u = \frac{c n \epsilon_0}{2} |E|^2$$

n is the refractive index of the medium, c is the speed of light in vacuum and ϵ_0 is the permittivity of free space. This formula assumes that the magnetic susceptibility is negligible, i.e. $\mu_r \approx 1$ where μ_r is the magnetic permeability of the light transmitting media. This assumption is typically valid in transparent media in the optical frequency range.

At a point Q in the aperture plane, the energy density is given by the symbol u_Q and in the observation plane at a point P the symbol used is u_P . The energy transferred U from the aperture plane to observation plane per second is found by integrating the energy density with respect to an area

$$(8a) \quad U = \int_S u dS \quad [\text{W or J/s}]$$

The energy per second radiated from the aperture is

$$(8b) \quad U_Q = \int_{S_Q} u_Q dS \quad \text{integration over the area of the aperture}$$

The energy per second reaching an area S_P of the observation plane is

$$(8c) \quad U_P = \int_{S_P} u_P dS \quad \begin{array}{l} \text{integration over the area element of the} \\ \text{observation plane} \end{array}$$

In the msript **op_rs_rxy_02.m** for the aperture, the dimensions a_x and a_y are specified [m] and the energy density u_Q [$\text{W}\cdot\text{m}^{-2}$]. Hence, the electric field E_Q within the aperture is

$$(9) \quad E_Q = \sqrt{\left(\frac{2u_Q}{cn\epsilon_0}\right)}$$

The integral to find the electric E_P at a point P in the observation plane is

$$(4) \quad E_P = \frac{1}{2\pi} \iint_{S_A} E_Q \frac{e^{jk r_{PQ}}}{r_{PQ}^3} z_P (jk r_{PQ} - 1) dS$$

where E_Q is given by equation (9) and the integral is computed using a two-dimensional form of Simposn's 1/3 rule.

The integrals given in equations (8) are also computed using the two-dimensional form of Simpson's 1/3 rule.

Once E_P is known, the energy density u_P is calculated from equation (7) and energy U_P from equation (8c).

SIMULATIONS `op_rs_rxy_01.m`

Rectangular aperture (far field)

Input parameters

```
nP = 495;           % observation points for P must be ODD
nQ = 147;           % aperture points for Q must be ODD

wL = 632.8e-9;      % wavelength [m]

aQx = 1e-4;         % defines aperture size [m]
aQy = 2e-4;

aP = 30e-3;         % defines size of observation space [m]

zQ = 0; zP = 1;    % z distance between aperture plan and
                    % observation plane [m]
```

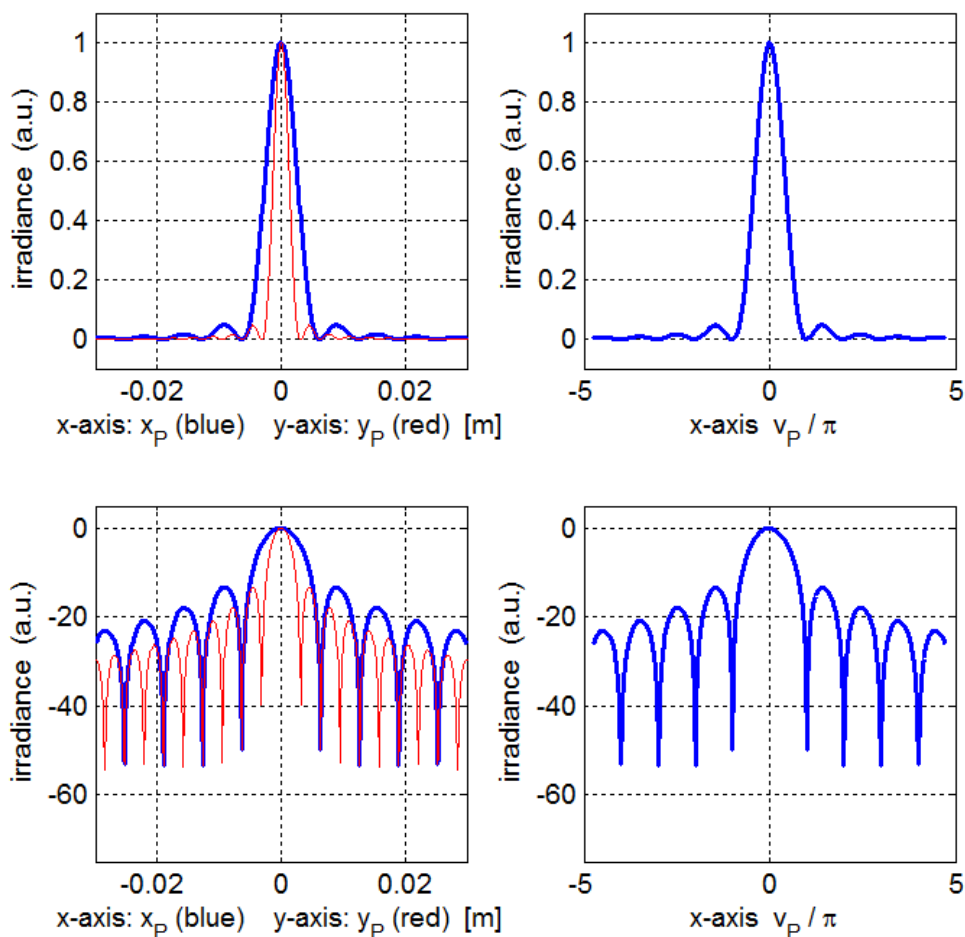


Fig. 2. Irradiance patterns in the along the X and Y axes. The maximum irradiance has been normalized to 1. The bottom curves use a log scale for the irradiance $I_{dB} = 10\log_{10}(I)$.

Figures (2) and (3) shows the irradiance for a rectangular aperture with $a_y = 2 a_x$. The diffraction pattern is characterized by a strong central maximum and very weak peaks of decreasing magnitude away from the optical axis. The separation of the dark bands along the Y axes is narrower than along the X axis. Figure (3) shows photograph like images of the diffraction pattern. The irradiance is scaled to represent the different exposure times.

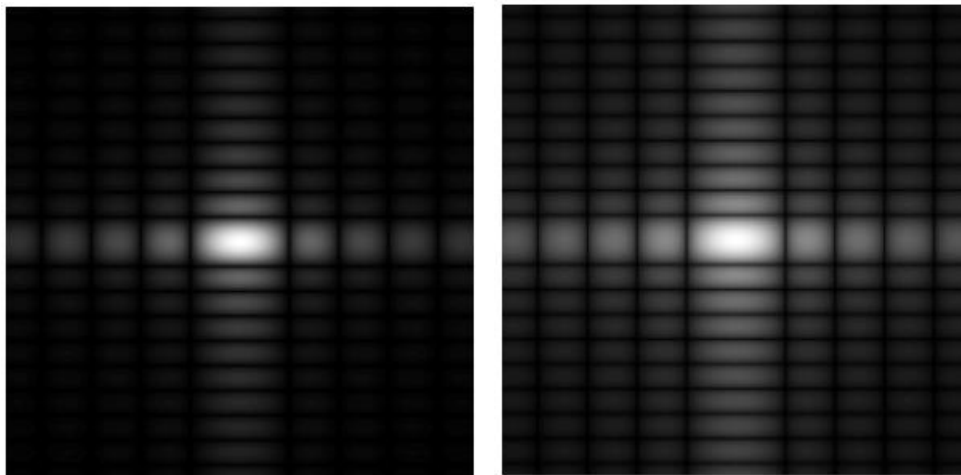


Fig. 3. Photograph like images for the diffraction patterns for a rectangular aperture. The image on the right is scaled to represent a longer exposure time.

In Tables 2 and 3 are listed values of the first few maxima and zeros of the diffraction pattern that were calculated using `op_rs_rxy_01.m` and the function `turningPoints.m`. The function returns the values of `indexMax` and `indexMin`. Then in the Command Window the values of the radial coordinates for the peaks and zeros can be displayed.

The theoretical values are for the zero crossing and minima and maxima of the function $y_m^2 = (\sin x / x)^2$ which gives the Fraunhofer diffraction pattern for a single slit.

Table 2. Relative irradiances of the maxima of the diffraction pattern.

peaks	central	1 st	2 nd	3 rd	4 th
theory	0	± 1.429	± 2.462	± 3.470	± 4.478
vP / π	0	± 1.440	± 2.457	± 3.473	± 4.470
theory	1.0000	0.0472	0.0165	0.0083	0.0050
I/I_0	1.0000	0.0471	0.0165	0.0083	0.0050

Table 3. Zeros in the irradiance for the diffraction pattern

zeros	1 st	2 nd	3 rd	4 th
theory	± 1	± 2	± 3	± 4
v_P / π	± 0.9981	± 1.9960	± 2.9937	± 3.9910

How finding the zero crossings (indexMin) and maxima (indexMax) in the **Command Window** where indexMin gives the position in an array for the zeros and indexMax gives the position in arrays for the maxima. Note the Matlab variable for the irradiance is **WP**.

Type in the Command Window:

```
yData = WP(indexyP,:);
xData = xP(indexyP,:);
[indexMin indexMax] = turningPoints(xData,yData);
```

```
indexMin → 24 55 86 117 179 210 241 272
```

```
vPx(indexyP, indexMin)./pi →
```

```
-3.9910 -2.9937 -1.9960 -0.9981 0.9981 1.9960 2.9937 3.9910
```

```
indexMax → 9 40 72 104 148 192 224 256 287
```

```
vPx(indexyP, indexMax)./pi →
```

```
-4.470 -3.473 -2.457 -1.4440 0.0000 1.4440 2.457 3.473 4.470
```

```
WP(indexyP, indexMax) →
```

```
0.0050 0.0083 0.0165 0.0471 1.0000 0.0471 0.0165 0.0083 0.0050
```

In a figure Window you can use the Data Cursor to find values for the zero crossings and maxima.

There is excellent agreement between the predictions of the Fraunhofer approximation and the calculations performed by evaluating the Rayleigh-Sommerfeld diffraction integral.

Diffraction pattern in near field

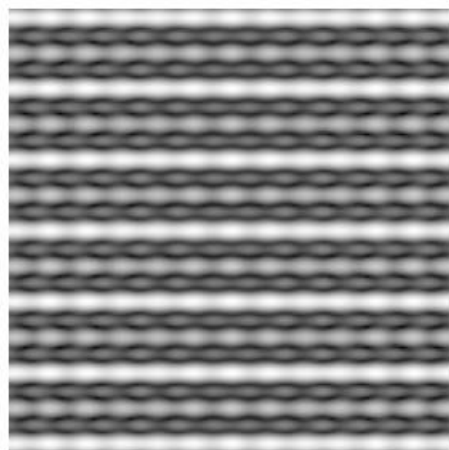
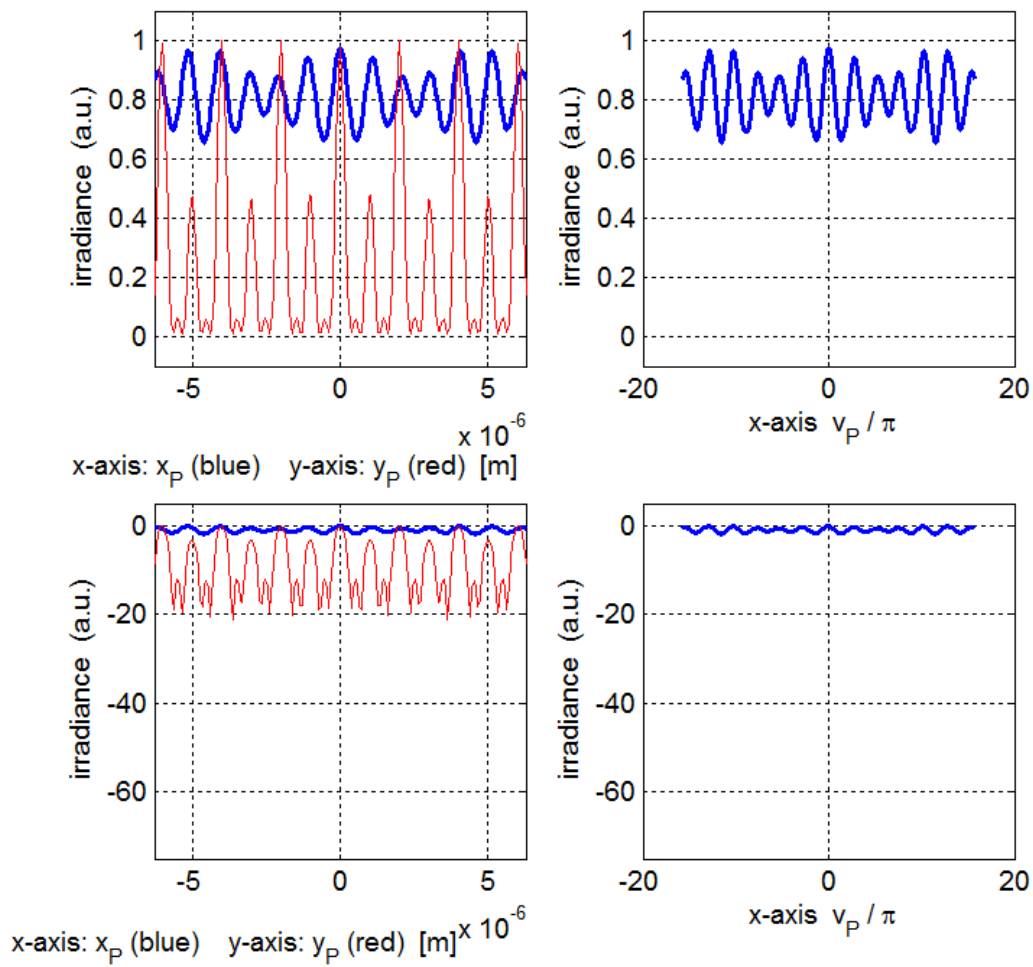


Fig. 4. Diffraction pattern from a rectangular aperture. Observation plane close to aperture plane $z_P = 100\lambda$ and observation plane dimensions $10\lambda \times 10\lambda$.

Diffraction – single slit

slit dimensions $a_x = 1 \times 10^{-4}$ m $a_y = 1 \times 10^{-6}$ m

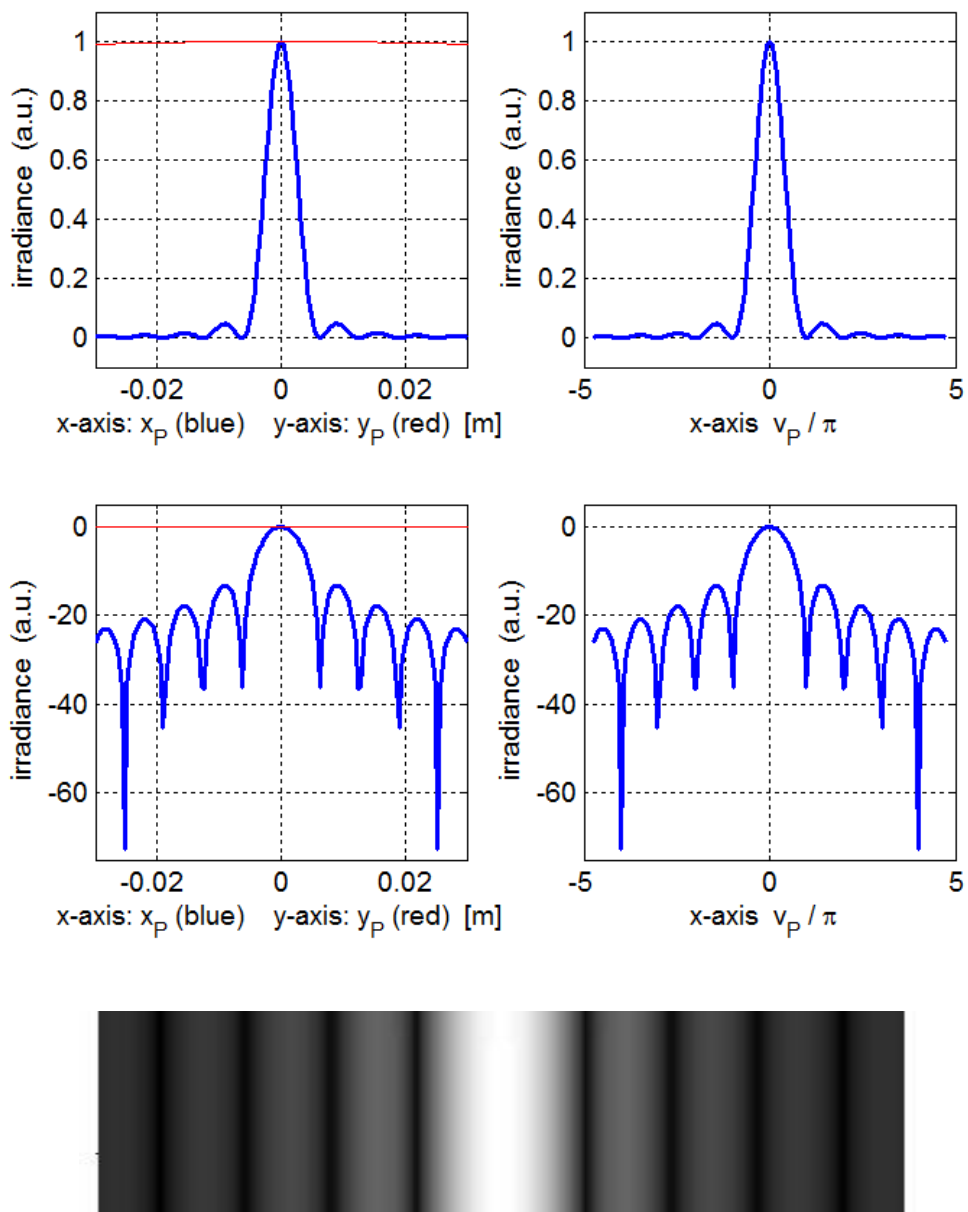


Fig. 5. Diffraction pattern from a single slit.

Diffraction – square aperture

slit dimensions $a_x = 1 \times 10^{-4}$ m $a_y = 1 \times 10^{-4}$ m

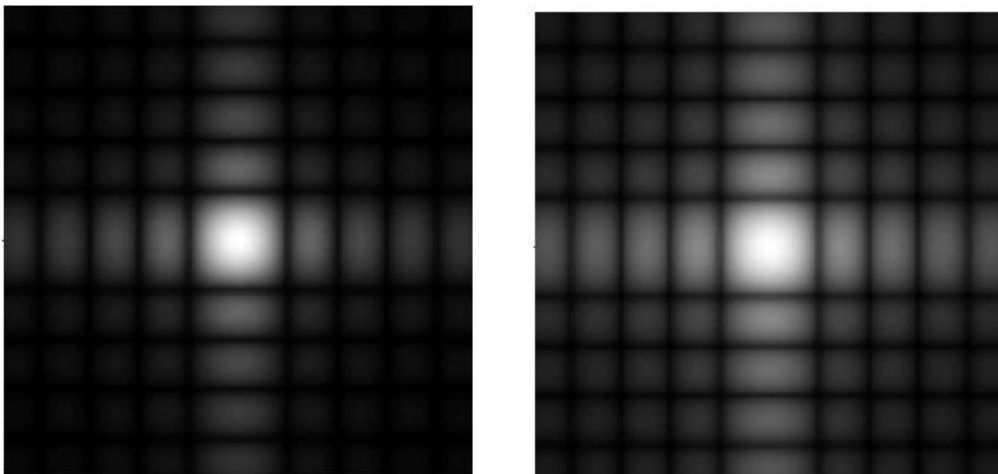
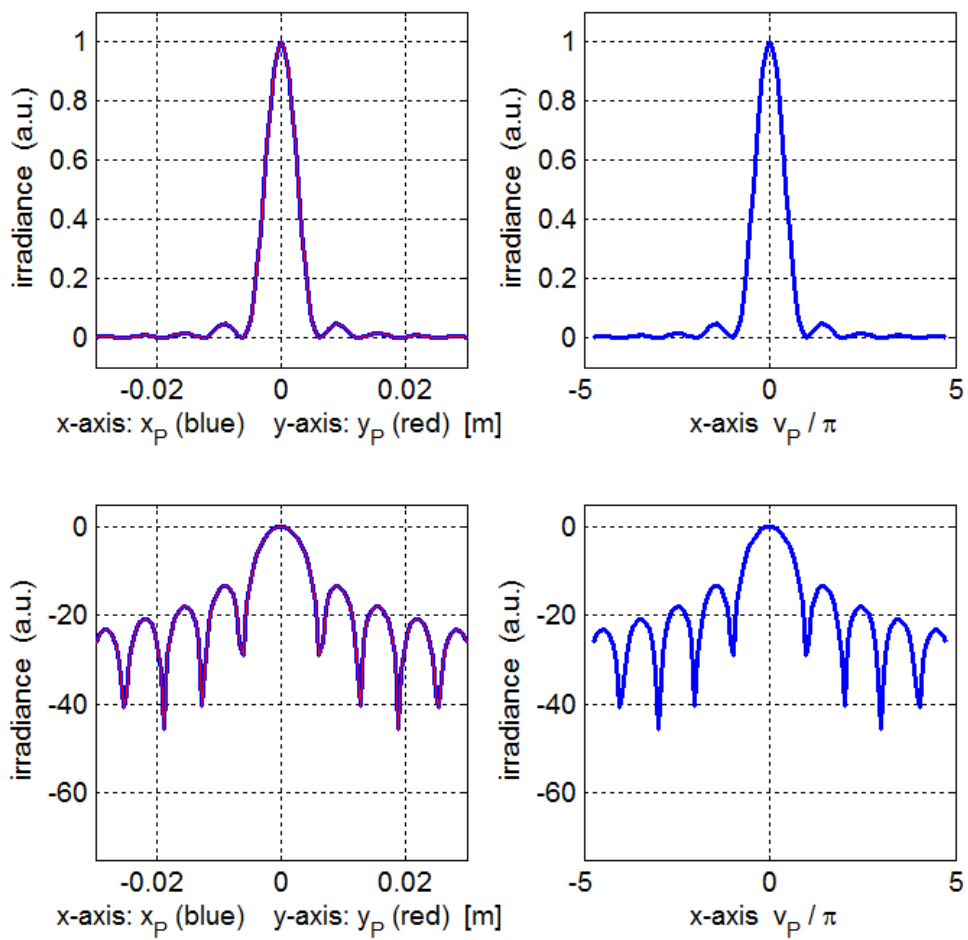


Fig. 6. Diffraction pattern from a square aperture with different scaling for the irradiance.

SIMULATIONS `op_rs_rxy_02.m`

The parameters for the aperture and observation planes are all changed within the mscript. After the program is run, a summary of the input and output parameters is shown in the Command Window.

Table 1: Parameter summary [SI units]

wavelength [m] = 6.33e-07

nQ = 099

nP = 201

Aperture Space

X width [m] = 1.000e-04

Y width [m] = 2.000e-04

energy density [W/m²] uQmax = 1.000e-03

energy from aperture [J/s] UQ(theory) = 2.000e-11

energy from aperture [J/s] UQ(calculated) = 2.000e-11

Observation Space

X width [m] = 4.000e-02

Y width [m] = 4.000e-02

distance aperture to observation plane [m] zP = 1.000e+00

Fraunhofer: position of 1st min in X direction [m] = 6.328e-03

Fraunhofer: position of 1st min in Y direction [m] = 3.164e-03

max energy density [W./m²] uPmax = 9.987e-07

energy to aperture [J/s] UP = 1.902e-11

percentage energy enclosed within circle of radius (vPx = 3.14159) = 99.5

Elapsed time is 40.470176 seconds.

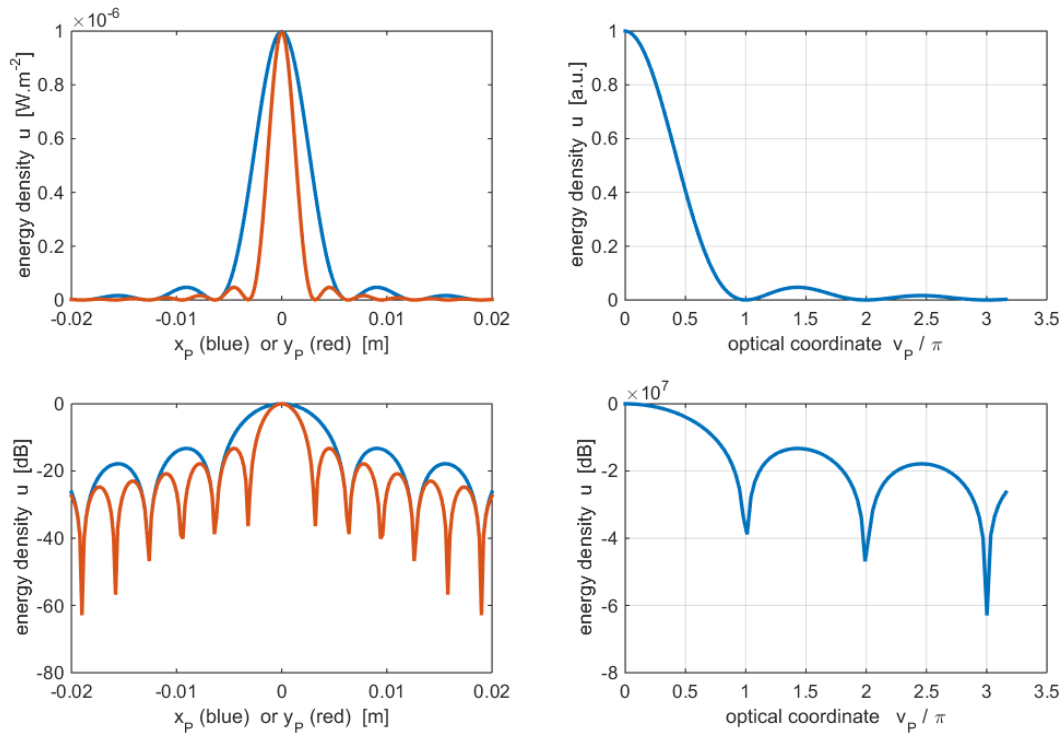


Fig. 7. The variation on the energy density along the X and Y directions. The plot was created in Matlab version 2014b which has improved graphics compared to earlier versions.

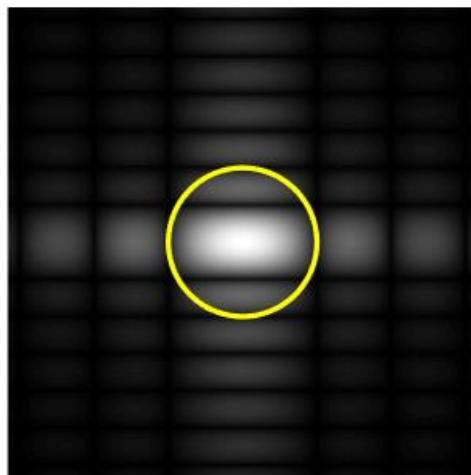


Fig. 8. Black and white photograph like time exposure of the diffraction pattern. The yellow circle has a radius equal to the position of the first zero along the X axis. 99.5% of the energy incident upon the observation screen is concentrated within this circle.

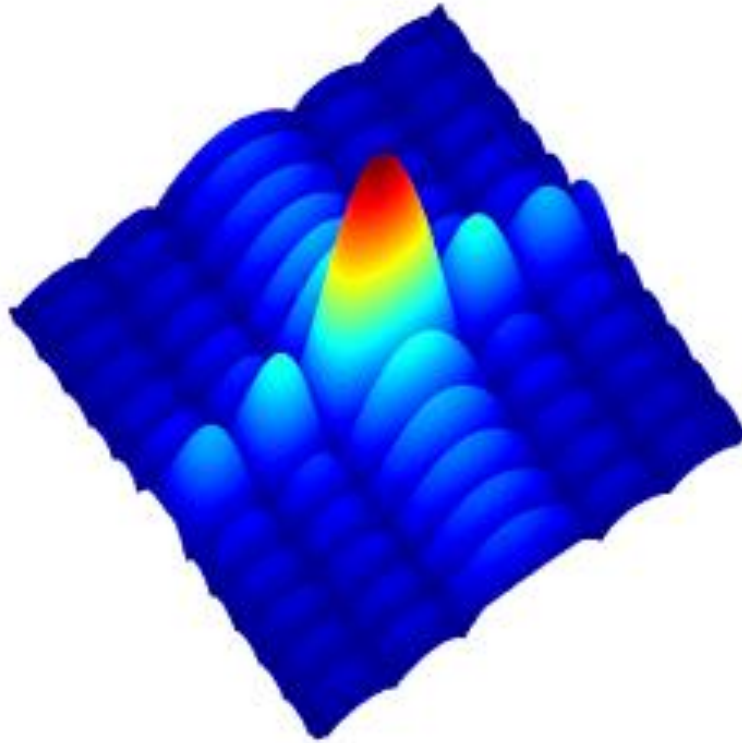


Fig. 9. A Matlab surf style plot showing the energy density in the XY observation plane.

Investigations

In the Fraunhofer region (far field) the **inverse square law** applies. This can be tested by doubling the aperture plane to the observation plane distance and noting that the maximum energy density at a point on the Z axis decreases by a factor of 4. For example using the data as shown in Table 1 and only varying the distance z_p :

$$z_{P1} = 1$$

$$\text{energy density } u_{P_{max1}} = 9.987 \times 10^{-7} \text{ W.m}^{-2}$$

$$1^{\text{st}} \text{ min } \quad x_{1\text{min}} = 6.328 \times 10^{-3} \text{ m} \quad y_{1\text{min}} = 3.164 \times 10^{-3} \text{ m}$$

$$z_{P2} = 2$$

$$\text{energy density } u_{P_{max2}} = 2.3497 \times 10^{-7} \text{ W.m}^{-2}$$

$$1^{\text{st}} \text{ min } \quad x_{2\text{min}} = 12.66 \times 10^{-3} \text{ m} \quad y_{2\text{min}} = 6.328 \times 10^{-3} \text{ m}$$

$$z_{P2} / z_{P1} = 2$$

$$u_{P_{max2}} / u_{P_{max1}} = 1/4 \quad \text{inverse square law satisfied}$$

$$x_{2\text{min}} / x_{1\text{min}} = 2.0 \quad y_{2\text{min}} / y_{1\text{min}} = 2.0 \quad \text{pattern is wider}$$

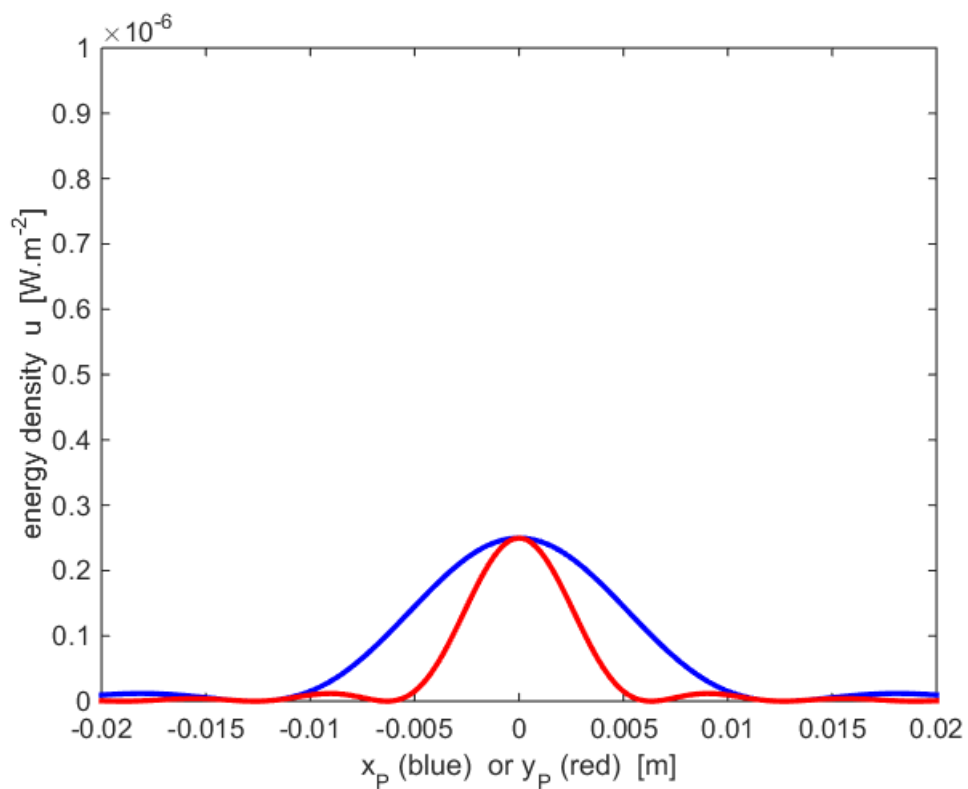
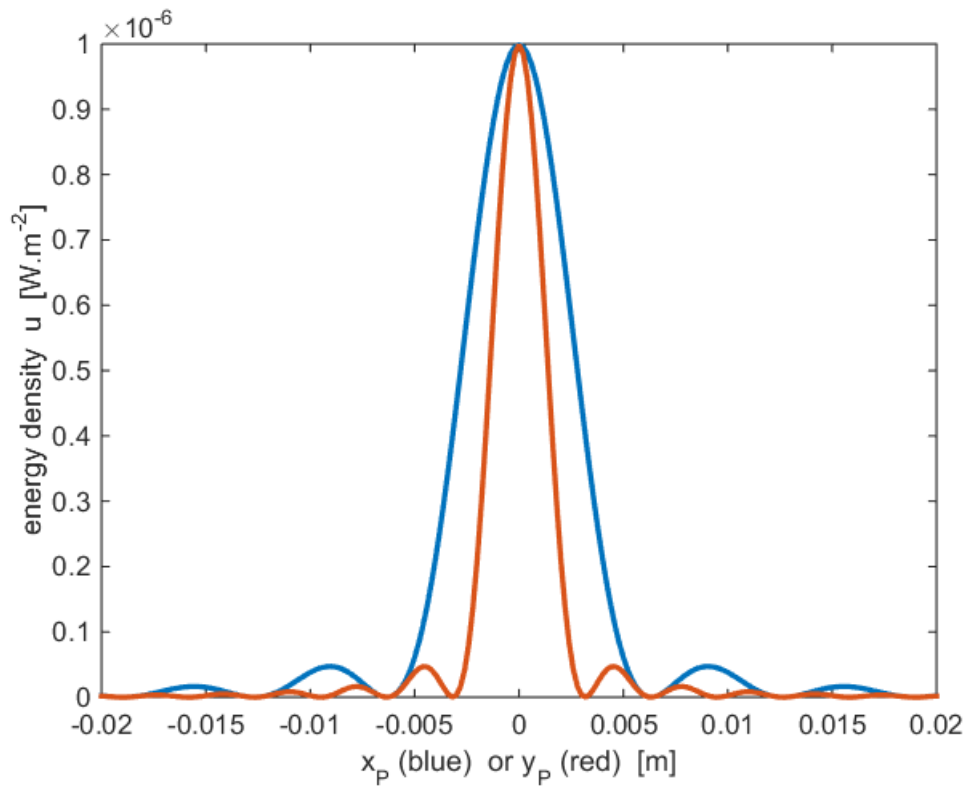


Fig. 10. Fraunhofer diffraction: upper plot $z_p = 1.0$ m and lower plot $z_p = 2.0$ m. As the aperture to observation plane distance increases the energy density distribution shape does not change but becomes wider and flatter.

Next, we can consider the effect of halving the area of the aperture by changing the width a_y in the Y direction from 2×10^{-4} m to 1.0×10^{-4} m.

$$\text{Width } a_{y1} = 2 \times 10^{-4} \text{ m} \quad a_{y2} = 1 \times 10^{-4} \text{ m}$$

Energy transferred from aperture to observation plane

$$U_{Q1} = 2.0 \times 10^{-11} \text{ J.s}^{-1} \quad U_{Q2} = 1.0 \times 10^{-11} \text{ J.s}^{-1}$$

$$U_{Q2} / U_{Q1} = 0.50$$

Maximum energy density in observation plane

$$u_{Pmax1} = 9.987 \times 10^{-7} \text{ W.m}^{-2} \quad u_{Pmax2} = 2.497 \times 10^{-7} \text{ W.m}^{-2}$$

$$u_{Pmax2} / u_{Pmax1} = 0.25$$

The effect of doubling the area of the aperture results in a doubling of the energy transferred from the aperture to the observation plane and a four-fold increase in the energy density.

Fresnel Diffraction ↔ **Fraunhofer Diffraction**

Fraunhofer diffraction is only valid in the far field. There is a transition from Fraunhofer diffraction to Fresnel diffraction as the distance between the aperture and observation planes decreases. The distance dividing the two regimes is known as the

Rayleigh distance d_{RL}

$$d_{RL} = \frac{a^2}{\lambda} \quad \text{where } a \text{ is the maximum of } a_x \text{ and } a_y$$

Fraunhofer diffraction (far field) $z_P > d_{RL}$

Fresnel diffraction (near field) $z_P < d_{RL}$

For an aperture $a_x = a_y = 1.0 \times 10^{-4}$ m and wavelength $\lambda = 6.33 \times 10^{-7}$ m, the Rayleigh distance is

$$d_{RL} = 1.58 \times 10^{-2} \text{ m}$$

The msript **op_rs_rxy_02.m** can be run a few times to investigate the transition regime when $z_P \sim d_{RL}$ by changing the values for z_P , x_P and y_P in the msript.

Rayleigh distance $d_{RL} = 0.0158$ m

