DOING PHYSICS WITH PYTHON

RAYLEIGH-SOMMERFELD DIFFRACTION

RECTANGULAR APERTURES

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Please email any corrections, comments, suggestions or additions: **matlabvisualphysics@gmail.com**

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

op005.py (Fraunhofer) op005S.py (slit)

op005f.py (Fresnel) op005Z.py (Z-axis)

Calculation of the irradiance in a plane perpendicular to the optical axis for a uniformly illuminated rectangular aperture.

FRAUNHOFER DIFFRACTION – RECTANGULAR APERTURE

We will consider the Fraunhofer diffraction patterns for a single rectangular aperture that is uniformly illuminated. The X and Y widths for the rectangular aperture are a_x and a_y respectively. The irradiance distribution I is given by the product of two single slit functions

Fraunhofer diffraction

(1) $$I = I_o \left(\frac{\sin(v_{Px})}{v_{Px}}\right)^2 \left(\frac{\sin(v_{Py})}{v_{Py}}\right)^2$$

where I_0 is a normalizing constant and the optical coordinates v_{Px} and v_{Py} are

(2) $$v_{Px} = \tfrac{1}{2} k\, a_x \sin\theta_x = \left(\frac{\pi}{\lambda}\right) a_x \sin\theta_x$$
$$v_{Py} = \tfrac{1}{2} k\, a_y \sin\theta_y = \left(\frac{\pi}{\lambda}\right) a_y \sin\theta_y$$

and the angles θ_x and θ_y define the direction of the diffracted ray

(3) $$\sin\theta_x = \frac{x_P}{\sqrt{x_P{}^2 + z_P{}^2}} \qquad \sin\theta_y = \frac{y_P}{\sqrt{y_P{}^2 + z_P{}^2}}$$

The geometry for the diffraction pattern from a rectangular aperture is shown in figure (1).

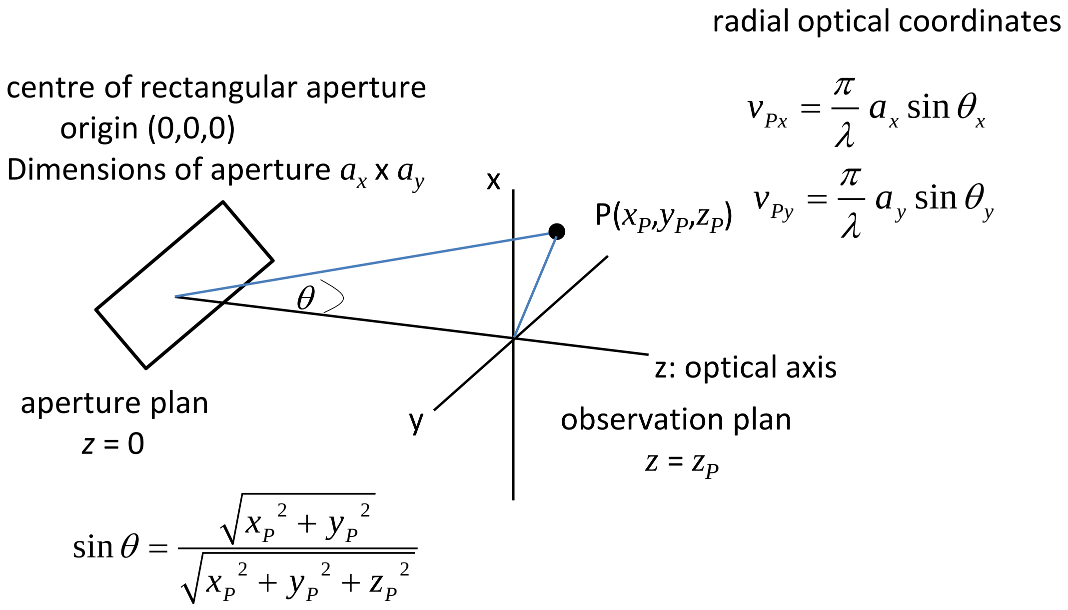


Fig. 1. Geometry for a rectangular aperture with dimensions a_x x a_y. The radial coordinates are scaled perpendicular distance from the optical axis.

The resulting diffraction pattern for the irradiance has lines of zeros when

(3a)
$$v_{Px} = m_x \pi \quad m_x = \pm 1, \pm 2, \pm 3, \ldots$$
$$v_{Py} = m_y \pi \quad m_y = \pm 1, \pm 2, \pm 3, \ldots$$

or

(3b) $$m_x \lambda = a_x \sin\theta_x \qquad m_y \lambda = a_y \sin\theta_y$$

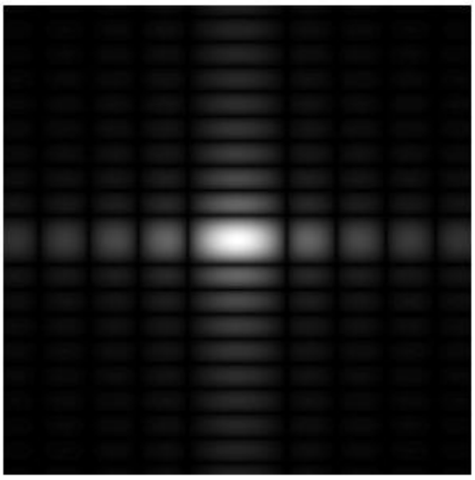


Fig. 2. Photograph like image for the Fraunhofer diffraction pattern of a rectangular aperture $a_y = 2\,a_x$.

However, a more versatile approach to study the diffraction from an aperture is to evaluate the **Rayleigh-Sommerfeld diffraction integral of the first kind**. No approximations need to be made and the irradiance can be calculated in the near and far fields in an XY plane and along the Z axis right up to the aperture. There is excellent agreement between the far field predictions of the Fraunhofer theory and the results of evaluating the Rayleigh-Sommerfeld diffraction integral.

RAYLEIGH DIFFRACTION INTEGRAL OF THE FIRST KIND

The **Rayleigh-Sommerfeld region** includes the entire space to the right of the aperture. It is assumed that the Rayleigh-Sommerfeld diffraction integral of the first kind is valid throughout this space, right down to the aperture. There are no limitations on the maximum size of either the aperture or observation region, relative to the observation distance, because **no approximations have been made**.

The Rayleigh-Sommerfeld diffraction integral of the first kind (RS1) can be expressed as

$$E_P = \frac{1}{2\pi} \iint_{S_A} E_Q \frac{e^{j k r_{PQ}}}{{r_{PQ}}^3} z_p \,(j k r_{PQ} - 1)\, \mathrm{d}S \tag{4}$$

where E_P is the electric field at the observation point P, E_Q is the electric field within the aperture and r_{PQ} is the distance from an aperture point Q to the observation point P. The double integral is over the area of the aperture S_A. The wavelength of the light and the propagation constant are $\lambda = \frac{2\pi}{k} \quad k = \frac{2\pi}{\lambda}$.

Evaluating The Diffraction Integral using **op005.py**

The double integral can be estimate numerically by a two-dimensional form of Simpson's 1/3 rule. The electric field E_P at the point P is computed by

$$(5) \qquad E_P = \sum_{m=1}^{N}\sum_{n=1}^{N}\left(E_{Qmn}\frac{e^{jkr_{PQmn}}}{r_{PQmn}^{\;3}} z_{Pmn}\left(jk\,r_{PQmn} - 1\right)\right)\left(\frac{h_x h_y}{9} S_{mn}\right)$$

$$h_x = \frac{b_x - a_x}{N-1} \qquad h_y = \frac{b_y - a_y}{N-1}$$

where S_{mn} are the Simpson's two-dimensional coefficients, (a_x, b_x) and (a_y, b_y) are the lower and upper bounds of the aperture and E_0 is a normalizing constant. Each term in equation (2) can be expressed as a matrix of size $N \times N$ and the matrices can be manipulated very easily in Python to give the estimate of the integral. The irradiance is proportional to the square of the magnitude of the electric field, hence the irradiance in the space beyond the aperture can be calculated by

$$(6) \qquad I = I_0 \left| E^* E \right|$$

where I_0 is a normalizing constant and E^* is the complex conjugate of E.

We can simply equation (5) when the electric field and is calculated in arbitrary units and the irradiance is normalised so that the maximum value is one.

$$(6) \quad E_P = \sum_{m=1}^{N}\sum_{n=1}^{N}\left(\frac{e^{jkr_{PQmn}}}{r_{PQmn}{}^{3}}\left(jk\,r_{PQmn}-1\right)S_{mn}\right)$$

SIMULATIONS

Fraunhofer diffraction (far field) op005.py

wavelength = 6.328e-07 m

Aperature space

Grid point nQ = 201

X width = 2.00e-04 m

Y width = 4.00e-04 m

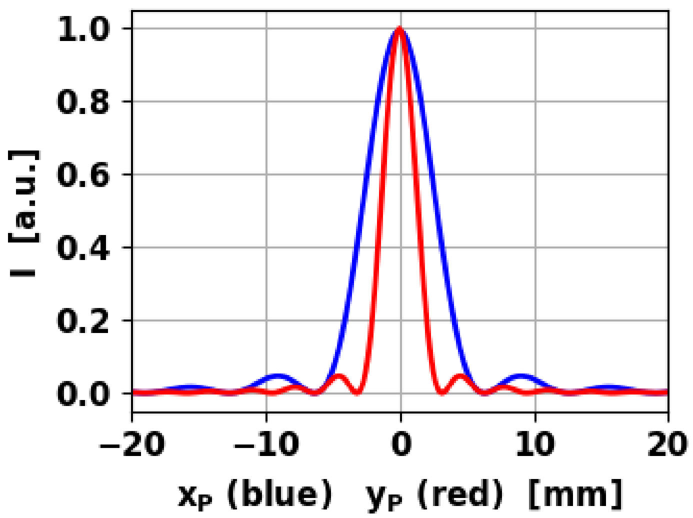
Observation

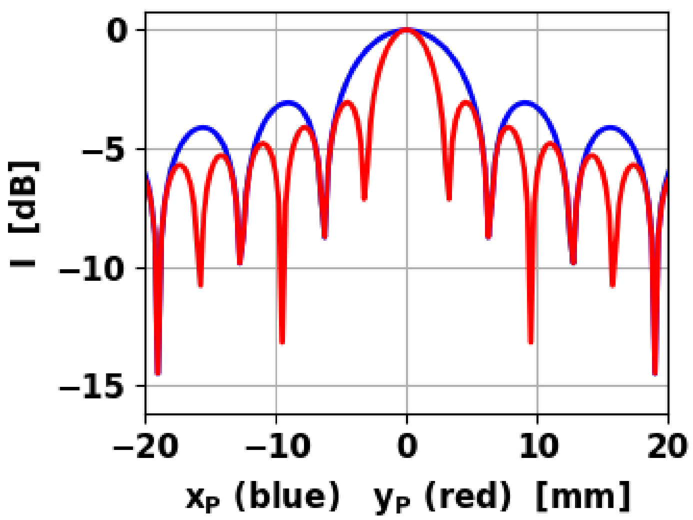
Grid point nP = 241

zP = 1.000 m

First min: x_0 = 6.328 mm

First min: y_0 = 3.164 mm


1.0
0.8
0.6
0.4
0.2
0.0
I [a.u.]
−20
−10
0
10
20
x_P (blue) y_P (red) [mm]


0
−5
−10
−15
I [dB]
−20
−10
0
10
20
x_P (blue) y_P (red) [mm]

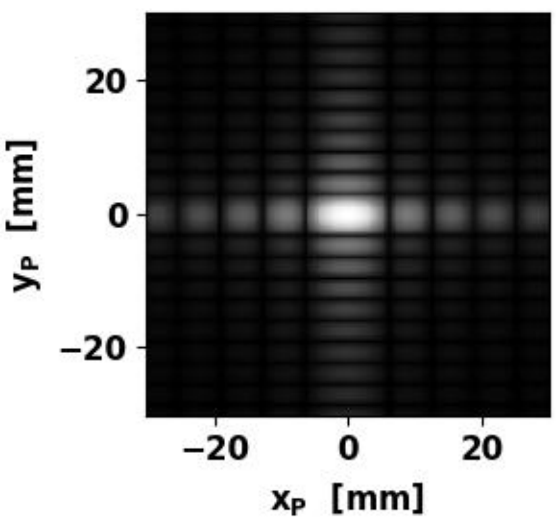

20
0
−20
y_P [mm]
−20
0
20
x_P [mm]

Fig. 3. Rectangular aperture diffraction. The positions of the first minima in the diffraction pattern agrees with the Fraunhofer predictions.

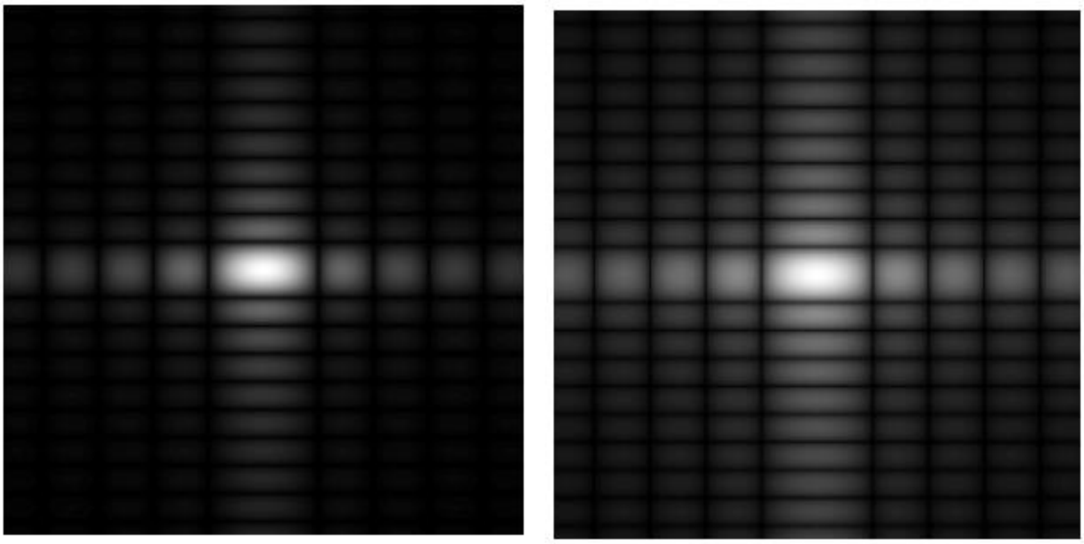


Fig. 4. Photograph like images of the diffraction pattern. The irradiance is scaled to represent the different exposure times.

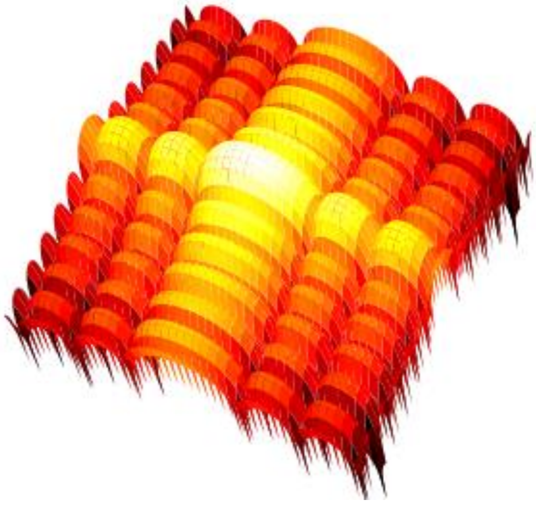


Fig. 5. [3D] plot of the irradiance [dB] for a rectangular aperture.

Figures 3 to 5 show the irradiance for a rectangular aperture with $a_y = 2\ a_x$. The diffraction pattern is characterized by a strong central maximum and very weak peaks of decreasing magnitude away from the optical axis. The separation of the dark bands along the *Y* axes are narrower than along the *X* axis.

There is excellent agreement between the predictions of the Fraunhofer approximation and the calculations performed by evaluating the Rayleigh-Sommerfeld diffraction integral in the far field.

Diffraction – single slit **op005S.py**

wavelength = 6.328e-07 m

Aperature space

Grid point nQ = 199

X width = 2.00e-04 m

Y width = 2.00e-06 m

Observation

Grid point nP = 201

zP = 1.000 m

First min: x_0 = 6.328 mm

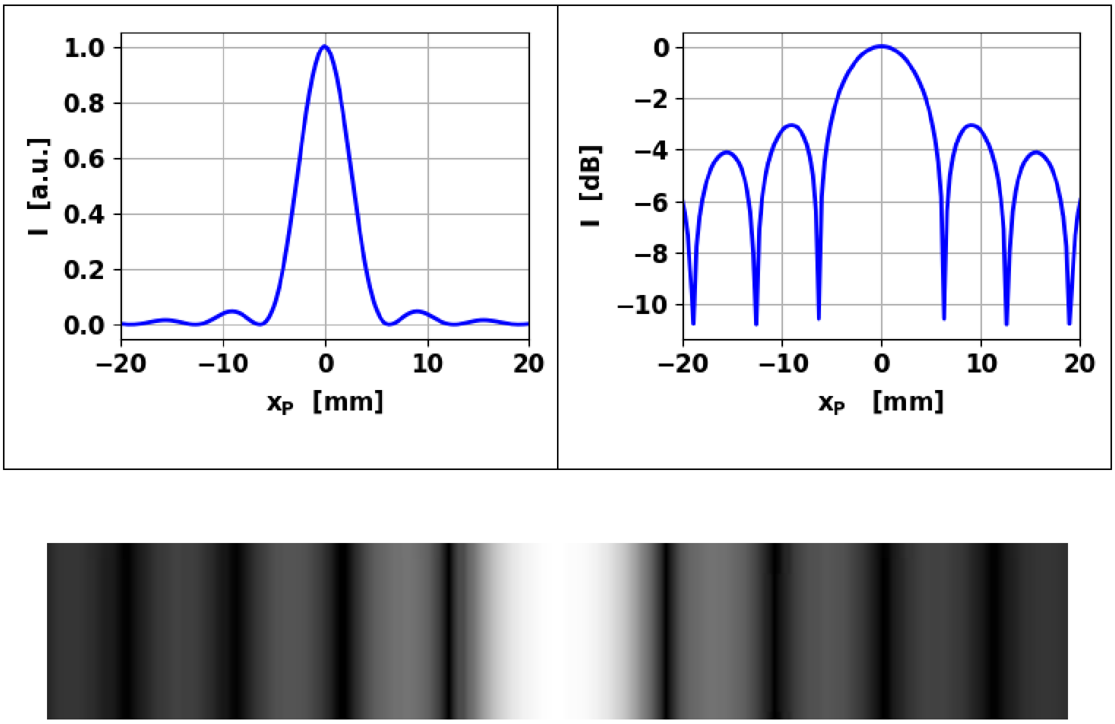


Fig. 5. Single slit diffraction simulation.

Fresnel diffraction pattern in near field   op005F.py

Fresnel diffraction occurs when either the distance from the source to the obstruction or the distance from the obstruction to the screen is comparable to the size of the obstruction. These comparable distances and sizes lead to unique diffractive behaviour. The approximations made for Fraunhofer Diffraction are no longer valid.

wavelength = 6.328e-07 m

Aperature space

Grid point nQ = 199

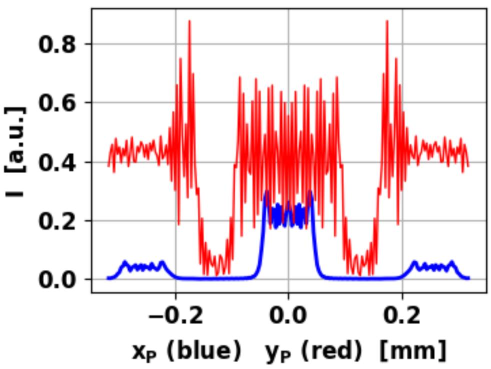
X width = 2.00e-04 m

Y width = 4.00e-04 m

Observation

Grid point nP = 201

zP = 3.164e-04



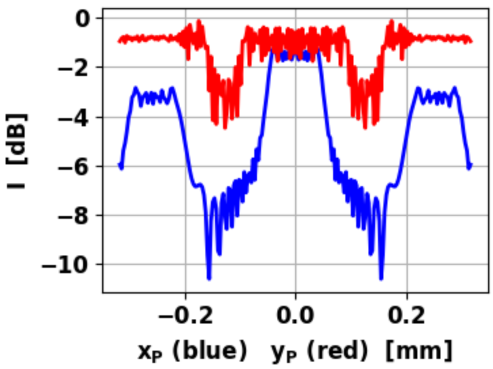
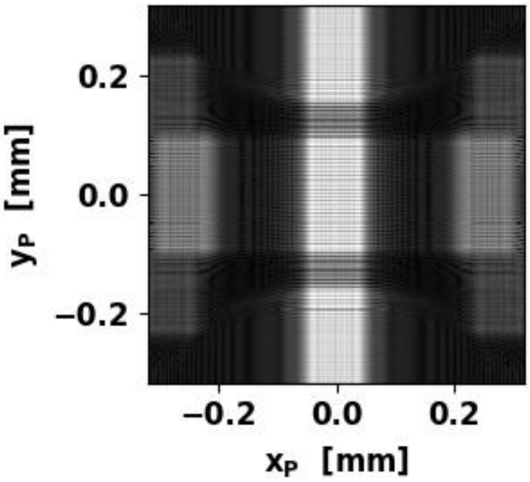


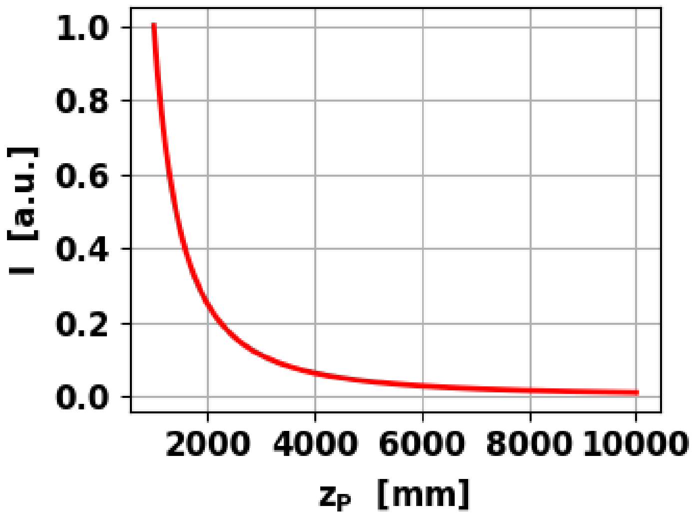
Fig. 6. Diffraction pattern from a rectangular aperture. Observation plane close to aperture plane $z_P = 500\lambda$.



Irradiance along the Z-axis op005Z.py

The Rayleigh-Sommerfeld integral can be used to calculate the irradiance along the Z-axis from the plane of the aperture.

For large distances from the aperture plane along the Z-axis, the aperture acts like a point source and the irradiance decreases according to the inverse square law (figure 7).



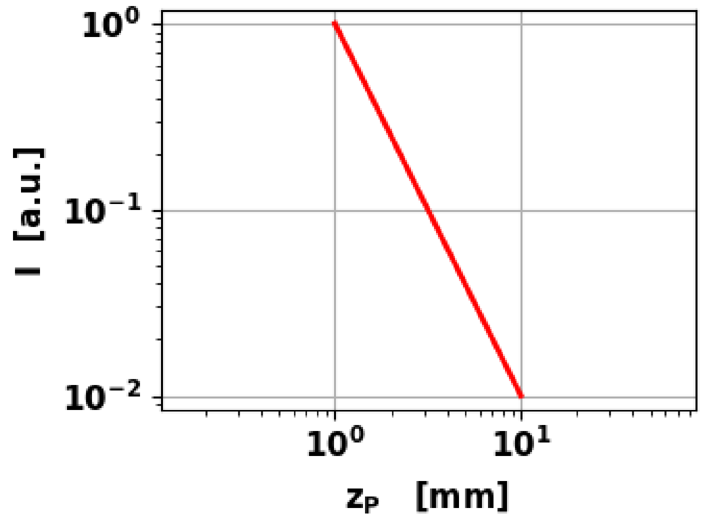


Fig. 7. The slope of the log-log plot of irradiance against the z distance is -2 which indicates that the irradiance falls off as a function of $1/z_P^2$.

The irradiance in the Z-direction shows wild fluctuates in the irradiance close to the aperture (figure 6).

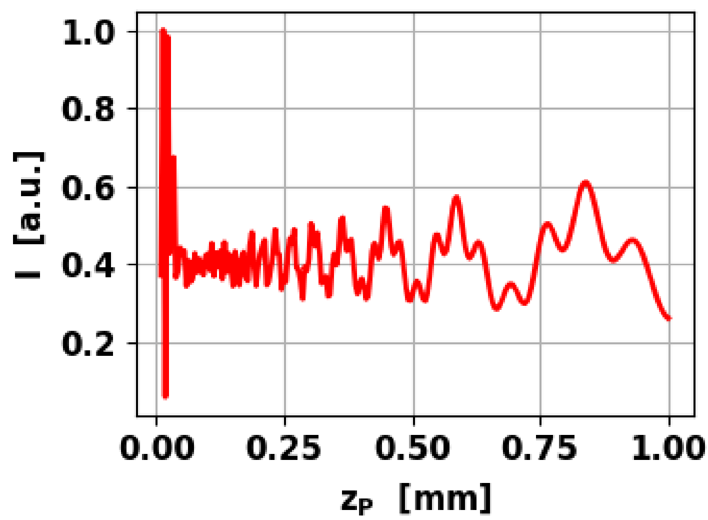


Fig. 8. Along the Z-axis, the irradiance fluctuates wildly near the aperture.